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Human Capital and International Real Business Cycles

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Technical Appendix

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1 Basic structure of the model.

- *Intertemporal utility function:*

The representative household in country *i* maximizes the intertemporal utility function:

$$U_{it} = E_0 \sum_{s=t}^{\infty} \frac{\beta^{s-t}}{1-\delta} [C_{is}^{\tau}(1 - n_{is} - e_{is})^{1-\tau}]^{1-\delta} \ , \tag{1}$$

where i=1,2, β is the intertemporal discount factor, δ is the relative risk aversion coefficient, τ measures the impact of leisure on welfare, C_{it} is *per capita* consumption, n_{it} is the time share devoted to consumption good production, and e_{it} is the time share devoted to human capital accumulation (each denominated in pure time units). We have $0<\beta<1$, $0<\tau<1$, $\sigma>0$, $C_{it}\geq 0$, $0\leq n_{it}\leq 1$, and $0\leq e_{it}\leq 1$. I assume the representative household is endowed with a fixed amount of time, normalized to 1 for simplicity.

- *Production function:*

Country *i* produces a homogenous consumption good with the following aggregate constant-returns-to-scale technology:

$$Y_{it} = A_{it}(v_{it}K_{it})^{1-\alpha}(n_{it}H_{it}^{\phi}\overline{H}_t^{1-\phi})^{\alpha} \ , \tag{2}$$

where v_{it} is the physical capital share devoted to the consumption good sector, K_{it} is the physical capital stock, and A_{it} is the sectoral total factor productivity (TFP). We have $0<\alpha<1$, $0<\phi<1$, $0\leq v_{it}\leq 1$, $Y_{it}\geq 0$, $K_{it}\geq 0$, $H_{it}\geq 0$. The representative household has accumulated through time a human capital stock equal to H_{it}. Human capital is interpreted as *knowledge capital* and is at least partially non-excludable. I suppose that households resident in one country can partially take advantage of human capital accumulated in the other country: formally, households in country *i* are able to sell an amount of effective labour equal to $H_{it}^{\phi}\overline{H}_t^{1-\phi}$, where $\overline{H}_t$ represents total (across countries) *per capita* stock of human capital; $\overline{H}_t$ is take as given by representative agents in both countries.

- *Physical capital accumulation technology:*

In country *i*, physical capital accumulates according to:

$$K_{it+1} = (1-\delta_K)K_{it} + \psi(\frac{X_{it}}{K_{it}})K_{it} \ . \tag{3}$$

where δ_K is the human capital depreciation rate, and the function $1/\psi'$ is equivalent to Tobin's q. We have $0<\delta_K<1$, and $X_{it}\geq 0$. I assume that near the steady-state $\psi>0$, $\psi'>0$ and $\psi''<0$; in particular, since I want the model with adjustment costs to perform near the steady state as the one without, I suppose that, in steady state, $\psi(\cdot) = X_i/K_i$ and $\psi'(\cdot) = 1$. Let us define the inverse of the adjustment cost elasticity as:

$$\xi_\psi \equiv -\frac{\psi''(\cdot)}{\psi'(\cdot)^2}\frac{X_i}{K_i} = -\psi''(\cdot)\frac{X_i}{K_i} \ . \tag{4}$$

- *Human capital accumulation technology:*

Human capital accumulates according to:

$$H_{it+1} = (1-\delta_H)H_{it} + B_{it}[(1-v_{it})K_{it}]^{1-\eta}(e_{it}H_{it}^{\phi}\overline{H}_t^{1-\phi})^{\eta} \ , \tag{5}$$

where δ_H is the human capital depreciation rate, and B_{it} is the sectoral TFP. We have $0<\eta<1$, and $0<\delta_H<1$.

- *Resource constraints:*

If Π_i is the fraction of world population that lives in country i, we can write the global resource constraint as:

$$\Pi_1(Y_{1t} - C_{1t} - X_{1t}) + \Pi_2(Y_{2t} - C_{2t} - X_{2t}) = 0 \ , \tag{6}$$

where X_{it} are investments in physical capital. We have $0<\Pi_1<1$ and $\Pi_2 = 1-\Pi_1$. For the sake of computational simplicity, lets denote NX_t the net trade of country 1 and rewrite the resource constraint in two separate equations:

$$\Pi_1(Y_{1t} - C_{1t} - X_{1t} - NX_t) = 0 \ , \qquad \Pi_2(Y_{2t} - C_{2t} - X_{2t} + \pi NX_t) = 0 \ , \tag{7}$$

where $\pi \equiv \Pi_1/\Pi_2$.

- *Stochastic structure:*

I assume that the stochastic processes driving TFP may be represented by the following first-order vector autoregression:

$$\ln(Z_{t+1}) = (I_4 - \rho)\ln(Z) + \rho\ln(Z_t) + \varepsilon_t \ , \qquad (8)$$

with $\varepsilon_t \sim N(0,\Sigma)$, where $Z_t \equiv [A_{1t} A_{2t} B_{1t} B_{2t}]^T$ and $\varepsilon_t \equiv [\epsilon_{1t}\, \epsilon_{2t}\, \nu_{1t}\, \nu_{2t}]^T$.

2 The optimization problem.

To solve the model I apply the procedure developed by King, Plosser and Rebelo (1987): in other words, I consider a deterministic version of the model, find out the first order conditions, normalize the system to make it stationary, log-linearize the first order conditions around the steady-state, add expectations to the linearized system, and solve it as a linear system of stochastic difference equations.

The presence of an externality implies that a dynamic competitive equilibrium has to be computed in two steps. As a first step, I substitute (2) into (7), solve (7) for C_{it}, and substitute everything into (1); then, exploiting the complete markets assumption I apply the Negishi-Mantel algorithm and solve a pseudo-planning problem for the representative agent, maximizing $\Pi U_1 + (1-\Pi)U_2$ subject to (3) and (5), and taking the arbitrary sequence $\{\overline{H}_t\}_0^\infty$ as given. The Lagrangian has the following form:

$$\begin{aligned} L = \sum_{t=0}^{\infty} \beta^t \Bigg[& \Pi_1 \frac{(A_{1t} v_{1t}^{1-\alpha} K_{1t}^{1-\alpha} n_{1t}^{\alpha} H_{1t}^{\alpha\phi} \overline{H}_t^{\alpha(1-\phi)} - X_{1t} - NX_t)^{\tau(1-\delta)} (1 - n_{1t} - e_{1t})^{(1-\tau)(1-\delta)}}{1-\delta} + \ldots \\ & \ldots + \Pi_2 \frac{(A_{2t} v_{2t}^{1-\alpha} K_{2t}^{1-\alpha} n_{2t}^{\alpha} H_{2t}^{\alpha\phi} \overline{H}_t^{\alpha(1-\phi)} - X_{2t} + \pi NX_t)^{\tau(1-\delta)} (1 - n_{2t} - e_{2t})^{(1-\tau)(1-\delta)}}{1-\delta} + \ldots \\ & \ldots + \sum_{i=1}^{2} \lambda_{it} \left[(1-\delta_K) K_{it} + \psi\left(\frac{X_{it}}{K_{it}}\right) K_{it} - K_{it+1} \right] + \ldots \\ & \ldots + \sum_{i=1}^{2} \mu_{it} \left[(1-\delta_H) H_{it} + B_{it} (1-v_{it})^{1-\eta} K_{it}^{1-\eta} e_{it}^{\eta} H_{it}^{\eta\phi} \overline{H}_t^{\eta(1-\phi)} - H_{it+1} \right] \Bigg] , \end{aligned} \qquad (9)$$

where $\lambda_{it} \equiv \lambda_{it}^* / \beta^t$ and $\mu_{it} \equiv \mu_{it}^* / \beta^t$. To obtain the first order conditions, I partially derive the Lagrangian with respect to $n_{it}, e_{it}, v_{it}, X_{it}, NX_t, K_{it+1}, H_{it+1}, \lambda_{it}$ and μ_{it} (deriving with respect to NX_t simply means to equalize the marginal utility of consumption in both countries). We end up with a system of 17 equations (for 8 control variables, one *control-like* variable, NX_t, and 8 state-costate variables). Finally, I impose the aggregate consistency condition $\overline{H}_t = H_{1t} + H_{2t}$.

3 First order conditions.

a) w.r.t. n_{it}:

$$\tau(1-n_{1t}-e_{1t})\alpha A_{1t}v_{1t}^{1-\alpha}K_{1t}^{1-\alpha}n_{1t}^{\alpha-1}H_{1t}^{\alpha\phi}H_t^{\alpha(1-\phi)} - (1-\tau)(A_{1t}v_{1t}^{1-\alpha}K_{1t}^{1-\alpha}n_{1t}^{\alpha}H_{1t}^{\alpha\phi}H_t^{\alpha(1-\phi)} - X_{1t} - NX_t) = 0 \quad (10)$$

$$\tau(1-n_{2t}-e_{2t})\alpha A_{2t}v_{2t}^{1-\alpha}K_{2t}^{1-\alpha}n_{2t}^{\alpha-1}H_{2t}^{\alpha\phi}H_t^{\alpha(1-\phi)} - (1-\tau)(A_{2t}v_{2t}^{1-\alpha}K_{2t}^{1-\alpha}n_{2t}^{\alpha}H_{2t}^{\alpha\phi}H_t^{\alpha(1-\phi)} - X_{2t} + \pi NX_t) = 0 \quad (11)$$

b) w.r.t. e_{it}:

$$\begin{aligned} &-\Pi_1(1-\tau)(A_{1t}v_{1t}^{1-\alpha}K_{1t}^{1-\alpha}n_{1t}^{\alpha}H_{1t}^{\alpha\phi}H_t^{\alpha(1-\phi)} - X_{1t} - NX_t)^{\tau(1-\delta)}(1-n_{1t}-e_{1t})^{(1-\tau)(1-\delta)-1} + \ldots \\ &\qquad \ldots + \eta\mu_{1t}B_{1t}(1-v_{1t})^{1-\eta}K_{1t}^{1-\eta}e_{1t}^{\eta-1}H_{1t}^{\eta\phi}H_t^{\eta(1-\phi)} = 0 \ , \end{aligned} \quad (12)$$

$$\begin{aligned} &-\Pi_2(1-\tau)(A_{2t}v_{2t}^{1-\alpha}K_{2t}^{1-\alpha}n_{2t}^{\alpha}H_{2t}^{\alpha\phi}H_t^{\alpha(1-\phi)} - X_{2t} + \pi NX_t)^{\tau(1-\delta)}(1-n_{2t}-e_{2t})^{(1-\tau)(1-\delta)-1} + \ldots \\ &\qquad \ldots + \eta\mu_{2t}B_{2t}(1-v_{2t})^{1-\eta}K_{2t}^{1-\eta}e_{2t}^{\eta-1}H_{2t}^{\eta\phi}H_t^{\eta(1-\phi)} = 0 \ , \end{aligned} \quad (13)$$

c) w.r.t. v_{it}:

$$\begin{aligned} &\Pi_1\tau(A_{1t}v_{1t}^{1-\alpha}K_{1t}^{1-\alpha}n_{1t}^{\alpha}H_{1t}^{\alpha\phi}H_t^{\alpha(1-\phi)} - X_{1t} - NX_t)^{\tau(1-\delta)-1}(1-n_{1t}-e_{1t})^{(1-\tau)(1-\delta)}\times\ldots \\ &\ldots\times(1-\alpha)A_{1t}v_{1t}^{-\alpha}K_{1t}^{1-\alpha}n_{1t}^{\alpha}H_{1t}^{\alpha\phi}H_t^{\alpha(1-\phi)} - (1-\eta)\mu_{1t}B_{1t}(1-v_{1t})^{-\eta}K_{1t}^{1-\eta}e_{1t}^{\eta}H_{1t}^{\eta\phi}H_t^{\eta(1-\phi)} = 0 \ , \end{aligned} \quad (14)$$

$$\begin{aligned} &\Pi_2\tau(A_{2t}v_{2t}^{1-\alpha}K_{2t}^{1-\alpha}n_{2t}^{\alpha}H_{2t}^{\alpha\phi}H_t^{\alpha(1-\phi)} - X_{2t} + \pi NX_t)^{\tau(1-\delta)-1}(1-n_{2t}-e_{2t})^{(1-\tau)(1-\delta)}\times\ldots \\ &\ldots\times(1-\alpha)A_{2t}v_{2t}^{-\alpha}K_{2t}^{1-\alpha}n_{2t}^{\alpha}H_{2t}^{\alpha\phi}H_t^{\alpha(1-\phi)} - (1-\eta)\mu_{2t}B_{2t}(1-v_{2t})^{-\eta}K_{2t}^{1-\eta}e_{2t}^{\eta}H_{2t}^{\eta\phi}H_t^{\eta(1-\phi)} = 0 \ , \end{aligned} \quad (15)$$

d) w.r.t. X_{it}:

$$-\Pi_1\tau(A_{1t}v_{1t}^{1-\alpha}K_{1t}^{1-\alpha}n_{1t}^{\alpha}H_{1t}^{\alpha\phi}H_t^{\alpha(1-\phi)} - X_{1t} - NX_t)^{\tau(1-\delta)-1}(1-n_{1t}-e_{1t})^{(1-\tau)(1-\delta)} + \lambda_{1t}\psi'(\frac{X_{1t}}{K_{1t}}) = 0 \ , \quad (16)$$

$$-\Pi_2\tau(A_{2t}v_{2t}^{1-\alpha}K_{2t}^{1-\alpha}n_{2t}^{\alpha}H_{2t}^{\alpha\phi}H_t^{\alpha(1-\phi)} - X_{2t} + \pi NX_t)^{\tau(1-\delta)-1}(1-n_{2t}-e_{2t})^{(1-\tau)(1-\delta)} + \lambda_{2t}\psi'(\frac{X_{2t}}{K_{2t}}) = 0 \ , \quad (17)$$

d) w.r.t. NX$_t$:

$$
\begin{aligned}
&-\tau(A_{1t}v_{1t}^{1-\alpha}K_{1t}^{1-\alpha}n_{1t}^{\alpha}H_{1t}^{\alpha\phi}H_t^{\alpha(1-\phi)}-X_{1t}-NX_t)^{\tau(1-\delta)-1}(1-n_{1t}-e_{1t})^{(1-\tau)(1-\delta)}+\ldots\\
&\ldots+\tau(A_{2t}v_{2t}^{1-\alpha}K_{2t}^{1-\alpha}n_{2t}^{\alpha}H_{2t}^{\alpha\phi}H_t^{\alpha(1-\phi)}-X_{2t}+\pi NX_t)^{\tau(1-\delta)-1}(1-n_{2t}-e_{2t})^{(1-\tau)(1-\delta)}=0\ .
\end{aligned}
\tag{18}
$$

e) w.r.t. K$_{it+1}$:

$$
\begin{aligned}
&\beta^{t+1}\Pi_1\tau(A_{1t+1}v_{1t+1}^{1-\alpha}K_{1t+1}^{1-\alpha}n_{1t+1}^{\alpha}H_{1t+1}^{\alpha\phi}H_t^{\alpha(1-\phi)}-X_{1t+1}-NX_{t+1})^{\tau(1-\delta)-1}(1-n_{1t+1}-e_{1t+1})^{(1-\tau)(1-\delta)}\times\ldots\\
&\ldots\times(1-\alpha)A_{1t+1}v_{1t+1}^{1-\alpha}K_{1t+1}^{-\alpha}n_{1t+1}^{\alpha}H_{1t+1}^{\alpha\phi}H_t^{\alpha(1-\phi)}+\beta^{t+1}\lambda_{1t+1}\left[(1-\delta_K)+\psi(\frac{X_{1t+1}}{K_{1t+1}})-\psi'(\frac{X_{1t+1}}{K_{1t+1}})\frac{X_{1t+1}}{K_{1t+1}}\right]+\ldots\\
&\ldots+\beta^{t+1}\mu_{1t+1}(1-\eta)B_{1t+1}(1-v_{1t+1})^{1-\eta}K_{1t+1}^{-\eta}e_{1t+1}^{\eta}H_{1t+1}^{\eta\phi}H_t^{\eta(1-\phi)}-\lambda_{1t}\beta^t=0\ ,
\end{aligned}
\tag{19}
$$

$$
\begin{aligned}
&\beta^{t+1}\Pi_1\tau(A_{2t+1}v_{2t+1}^{1-\alpha}K_{2t+1}^{1-\alpha}n_{2t+1}^{\alpha}H_{2t+1}^{\alpha\phi}H_t^{\alpha(1-\phi)}-X_{2t+1}+\pi NX_{t+1})^{\tau(1-\delta)-1}(1-n_{2t+1}-e_{2t+1})^{(1-\tau)(1-\delta)}\times\ldots\\
&\ldots\times(1-\alpha)A_{2t+1}v_{2t+1}^{1-\alpha}K_{2t+1}^{-\alpha}n_{2t+1}^{\alpha}H_{2t+1}^{\alpha\phi}H_t^{\alpha(1-\phi)}+\beta^{t+1}\lambda_{2t+1}\left[(1-\delta_K)+\psi(\frac{X_{2t+1}}{K_{2t+1}})-\psi'(\frac{X_{2t+1}}{K_{2t+1}})\frac{X_{2t+1}}{K_{2t+1}}\right]+\ldots\\
&\ldots+\beta^{t+1}\mu_{2t+1}(1-\eta)B_{2t+1}(1-v_{2t+1})^{1-\eta}K_{2t+1}^{-\eta}e_{2t+1}^{\eta}H_{2t+1}^{\eta\phi}H_t^{\eta(1-\phi)}-\lambda_{2t}\beta^t=0\ ,
\end{aligned}
\tag{20}
$$

f) w.r.t. H$_{it+1}$:

$$
\begin{aligned}
&\beta^{t+1}\Pi_1\tau(A_{1t+1}v_{1t+1}^{1-\alpha}K_{1t+1}^{1-\alpha}n_{1t+1}^{\alpha}H_{1t+1}^{\alpha\phi}H_t^{\alpha(1-\phi)}-X_{1t+1}-NX_{t+1})^{\tau(1-\delta)-1}(1-n_{1t+1}-e_{1t+1})^{(1-\tau)(1-\delta)}\times\ldots\\
&\ldots\times\zeta A_{1t+1}v_{1t+1}^{1-\alpha}K_{1t+1}^{1-\alpha}n_{1t+1}^{\alpha}H_{1t+1}^{\alpha\phi-1}H_t^{\alpha(1-\phi)}+\mu_{1t+1}\beta^{t+1}(1-\delta_H)+\ldots\\
&\ldots+\chi\mu_{1t+1}\beta^{t+1}B_{1t+1}(1-v_{1t+1})^{1-\eta}K_{1t+1}^{1-\eta}e_{1t+1}^{\eta}H_{1t+1}^{\eta\phi-1}H_t^{\eta(1-\phi)}-\mu_{1t}\beta^t=0\ ,
\end{aligned}
\tag{21}
$$

$$
\begin{aligned}
&\beta^{t+1}\Pi_2\tau(A_{2t+1}v_{2t+1}^{1-\alpha}K_{2t+1}^{1-\alpha}n_{2t+1}^{\alpha}H_{2t+1}^{\alpha\phi}H_t^{\alpha(1-\phi)}-X_{2t+1}+\pi NX_{t+1})^{\tau(1-\delta)-1}(1-n_{2t+1}-e_{2t+1})^{(1-\tau)(1-\delta)}\times\ldots\\
&\ldots\times\zeta A_{2t+1}v_{2t+1}^{1-\alpha}K_{2t+1}^{1-\alpha}n_{2t+1}^{\alpha}H_{2t+1}^{\alpha\phi-1}H_t^{\alpha(1-\phi)}+\mu_{2t+1}\beta^{t+1}(1-\delta_H)+\ldots\\
&\ldots+\chi\mu_{2t+1}\beta^{t+1}B_{2t+1}(1-v_{2t+1})^{1-\eta}K_{2t+1}^{1-\eta}e_{2t+1}^{\eta}H_{2t+1}^{\eta\phi-1}H_t^{\eta(1-\phi)}-\mu_{2t}\beta^t=0\ ,
\end{aligned}
\tag{22}
$$

g) w.r.t. λ_{it}:

$$
(1-\delta_K)K_{it}+\psi(\frac{X_{it}}{K_{it}})K_{it}-K_{it+1}=0\ .
\tag{23}
$$

h) w.r.t. μ_{it}:

$$(1-\delta_H)H_{it} + B_{it}(1-v_{it})^{1-\eta}K_{it}^{1-\eta}e_{it}^{\eta}H_{it}^{\eta\phi}H_t^{\eta(1-\phi)} - H_{it+1} = 0 \; . \tag{24}$$

i) TVCs: $\lim_{t\to\infty} \beta^t \lambda_{it} K_{it} = 0$, $\lim_{t\to\infty} \beta^t \mu_{it} H_{it} = 0$

4 Normalized First Order Conditions.

Along a balanced growth path, the variables Y_i, C_i, X_i, K_i nd H_i will grow at the same rate γ, while n_i, v_i, and e_i stay constant. To perform any numerical simulation we need to transform the non-stationary model into a stationary one and linearly approximate it around its deterministic steady state. Divide both sides of (4) by H_{it} to get:

$$\frac{H_{it+1}}{H_{it}} = (1-\delta_H) + B_{it}\left((1-v_{it})\frac{K_{it}}{H_{it}}\right)^{1-\eta} e_{it}^{\eta}\left(1+\frac{H_{jt}}{H_{it}}\right)^{(1-\phi)\eta} . \tag{25}$$

where $i \neq j$. Since B_{it}, e_{it}, v_{it}, and K_{it}/H_{it} are stationary in steady-state, the growth rate of human capital in each country can be constant in steady-state only if H_{1t} and H_{2t} grow at the same rate. If $H_{1t+1}/H_{1t}=H_{2t+1}/H_{2t}$, then:

$$\frac{H_{1t+1}+H_{2t+1}}{H_{1t}+H_{2t}} = \frac{H_{1t+1}}{H_{1t}} = \frac{H_{2t+1}}{H_{2t}} \; . \tag{26}$$

To attain global stationarity, we normalize the model w.r.t. H_t, since (8) proves that in steady-state H_t grows at the same rate as human capital in both countries.

After the normalization we end up with a system of 10 control and *control-like* variables, 7 *state-like* and costate variables, and 4 exogenous state variables, losing one endogenous state variable (the normalized human capital stock in country 2, equal to 1 minus h_t, the normalized stock in country 1), and gaining one *control-like* variable (γ_t, the growth rate of H_t).

a) w.r.t. n_{it}:

$$\tau(1-n_{1t}-e_{1t})s_{ah}A_{1t}v_{1t}^{s_{ak}}k_{1t}^{s_{ak}}n_{1t}^{-s_{ak}}h_t^{\zeta} - (1-\tau)(A_{1t}v_{1t}^{s_{ak}}k_{1t}^{s_{ak}}n_{1t}^{s_{ah}}h_t^{\zeta} - x_{1t} - nx_t) = 0 \; , \tag{26}$$

$$\tau(1 - n_{2t} - e_{2t}) s_{ah} A_{2t} v_{2t}^{s_{ak}} k_{2t}^{s_{ak}} n_{2t}^{-s_{ak}} (1 - h_t)^{\zeta} - (1 - \tau)[A_{2t} v_{2t}^{s_{ak}} k_{2t}^{s_{ak}} n_{2t}^{s_{ah}} (1 - h_t)^{\zeta} - x_{2t} + \pi n x_t] = 0 \ , \tag{27}$$

where normalized variables are denoted by lowercase letters (while $h_t \equiv H_{1t}/H_t$), and $s_{ah} \equiv \alpha$, $s_{ak} \equiv 1 - \alpha$, $\zeta \equiv \alpha\phi$.

b) w.r.t. e_{it}:

$$-\Pi_1 (1 - \tau)(A_{1t} v_{1t}^{s_{ak}} k_{1t}^{s_{ak}} n_{1t}^{s_{ah}} h_t^{\zeta} - x_{1t} - n x_t)^{\xi_{lc}} (1 - n_{1t} - e_{1t})^{\xi_{ll}} + s_{bh} \tilde{\mu}_{1t} B_{1t} (1 - v_{1t})^{s_{bk}} k_{1t}^{s_{bk}} e_{1t}^{-s_{bk}} h_t^{\chi} = 0 \ , \tag{28}$$

$$-\Pi_2 (1 - \tau)[A_{2t} v_{2t}^{s_{ak}} k_{2t}^{s_{ak}} n_{2t}^{s_{ah}} (1 - h_t)^{\zeta} - x_{2t} + \pi n x_t]^{\xi_{lc}} (1 - n_{2t} - e_{2t})^{\xi_{ll}} + s_{bh} \tilde{\mu}_{2t} B_{2t} (1 - v_{2t})^{s_{bk}} k_{2t}^{s_{bk}} e_{2t}^{-s_{bk}} (1 - h_t)^{\chi} = \tag{29}$$

where $s_{bh} \equiv \eta$, $s_{bk} \equiv 1 - \eta$, $\xi_{lc} \equiv \tau(1 - \delta)$, $\xi_{ll} \equiv (1 - \tau)(1 - \delta) - 1$, $\chi \equiv \eta\phi$, and $\tilde{\mu}_{it} \equiv \mu_{it}/H_t^{\xi_{cc}}$.

c) w.r.t. v_{it}:

$$\begin{aligned} &\Pi_1 \tau (A_{1t} v_{1t}^{s_{ak}} k_{1t}^{s_{ak}} n_{1t}^{\alpha} h_t^{\zeta} - x_{1t} - n x_t)^{\xi_{cc}} (1 - n_{1t} - e_{1t})^{\xi_{cl}} s_{ak} A_{1t} v_{1t}^{-s_{ah}} k_{1t}^{s_{ak}} n_{1t}^{s_{ah}} h_t^{\zeta} + \ldots \\ &\qquad \ldots - s_{bk} \tilde{\mu}_{1t} B_{1t} (1 - v_{1t})^{-s_{bh}} k_{1t}^{s_{bk}} e_{1t}^{s_{bh}} h_t^{\chi} = 0 \ , \end{aligned} \tag{30}$$

$$\begin{aligned} &\Pi_2 \tau [A_{2t} v_{2t}^{s_{ak}} k_{2t}^{s_{ak}} n_{2t}^{\alpha} (1 - h_t)^{\zeta} - x_{2t} + \pi n x_t]^{\xi_{cc}} (1 - n_{2t} - e_{2t})^{\xi_{cl}} s_{ak} A_{2t} v_{2t}^{-s_{ah}} k_{2t}^{s_{ak}} n_{2t}^{s_{ah}} (1 - h_t)^{\zeta} + \ldots \\ &\qquad \ldots - s_{bk} \tilde{\mu}_{2t} B_{2t} (1 - v_{2t})^{-s_{bh}} k_{2t}^{s_{bk}} e_{2t}^{s_{bh}} (1 - h_t)^{\chi} = 0 \ , \end{aligned} \tag{31}$$

where $\xi_{cc} \equiv \tau(1 - \delta) - 1$, and $\xi_{cl} \equiv (1 - \tau)(1 - \delta)$.

d) w.r.t. X_{it}:

$$-\Pi_1 \tau (A_{1t} v_{1t}^{s_{ak}} k_{1t}^{s_{ak}} n_{1t}^{s_{ah}} h_t^{\zeta} - x_{1t} - n x_t)^{\xi_{cc}} (1 - n_{1t} - e_{1t})^{\xi_{cl}} + \tilde{\lambda}_{1t} \psi'\left(\frac{x_{1t}}{k_{1t}}\right) = 0 \ , \tag{32}$$

$$-\Pi_2 \tau [A_{2t} v_{2t}^{s_{ak}} k_{2t}^{s_{ak}} n_{2t}^{s_{ah}} (1 - h_t)^{\zeta} - x_{2t} + \pi n x_t]^{\xi_{cc}} (1 - n_{2t} - e_{2t})^{\xi_{cl}} + \tilde{\lambda}_{2t} \psi'\left(\frac{x_{2t}}{k_{2t}}\right) = 0 \ , \tag{33}$$

where $\tilde{\lambda}_{it} \equiv \lambda_{it}/H_t^{\xi_{cc}}$.

d) w.r.t. NX$_t$:

$$
\begin{aligned}
&-\tau(A_{1t}v_{1t}^{s_{ak}}k_{1t}^{s_{ak}}n_t^{\alpha}h_t^{\zeta}-x_{1t}-nx_t)^{\xi_{cc}}(1-n_{1t}-e_{1t})^{\xi_{cl}}+\ldots\\
&\ldots+\tau[A_{2t}v_{2t}^{s_{ak}}k_{2t}^{s_{ak}}n_{2t}^{s_{ah}}(1-h_t)^{\zeta}-x_{2t}+\pi nx_t]^{\xi_{cc}}(1-n_{2t}-e_{2t})^{\xi_{cl}}=0\ .
\end{aligned}
\tag{34}
$$

e) w.r.t. K$_{it+1}$:

$$
\begin{aligned}
&\Pi_1\tau(A_{1t+1}v_{1t+1}^{s_{ak}}k_{1t+1}^{s_{ak}}n_{1t+1}^{\alpha}h_{t+1}^{\zeta}-x_{1t+1}-nx_{t+1})^{\xi_{cc}}(1-n_{1t+1}-e_{1t+1})^{\xi_{cl}}\times\ldots\\
&\ldots\times s_{ak}A_{1t+1}v_{1t+1}^{s_{ak}}k_{1t+1}^{-s_{ah}}n_{1t+1}^{s_{ah}}h_{t+1}^{\zeta}+\tilde{\lambda}_{1t+1}\left[(1-\delta_K)+\psi(\frac{x_{1t+1}}{k_{1t+1}})-\psi'(\frac{x_{1t+1}}{k_{1t+1}})\frac{x_{1t+1}}{k_{1t+1}}\right]+\ldots\\
&\ldots+\tilde{\mu}_{1t+1}s_{bk}B_{1t+1}(1-v_{1t+1})^{s_{bk}}k_{1t+1}^{-s_{bh}}e_{1t+1}^{s_{bh}}h_{t+1}^{\chi}-\frac{\tilde{\lambda}_{1t}\gamma^{-\xi_{cc}}}{\beta}=0\ ,
\end{aligned}
\tag{35}
$$

$$
\begin{aligned}
&\Pi_2\tau[A_{2t+1}v_{2t+1}^{s_{ak}}k_{2t+1}^{s_{ak}}n_{2t+1}^{\alpha}(1-h_{t+1})^{\zeta}-x_{2t+1}+\pi nx_{t+1}]^{\xi_{cc}}(1-n_{2t+1}-e_{2t+1})^{\xi_{cl}}\times\ldots\\
&\ldots\times s_{ak}A_{2t+1}v_{2t+1}^{s_{ak}}k_{2t+1}^{-s_{ah}}n_{2t+1}^{s_{ah}}(1-h_{t+1})^{\zeta}+\tilde{\lambda}_{2t+1}\left[(1-\delta_K)+\psi(\frac{x_{2t+1}}{k_{2t+1}})-\psi'(\frac{x_{2t+1}}{k_{2t+1}})\frac{x_{2t+1}}{k_{2t+1}}\right]+\ldots\\
&\ldots+\tilde{\mu}_{2t+1}s_{bk}B_{2t+1}(1-v_{2t+1})^{s_{bk}}k_{2t+1}^{-s_{bh}}e_{2t+1}^{s_{bh}}(1-h_{t+1})^{\chi}-\frac{\tilde{\lambda}_{2t}\gamma^{-\xi_{cc}}}{\beta}=0\ ,
\end{aligned}
\tag{36}
$$

f) w.r.t. H$_{it+1}$:

$$
\begin{aligned}
&\Pi_1\tau(A_{1t+1}v_{1t+1}^{s_{ak}}k_{1t+1}^{s_{ak}}n_{1t+1}^{\alpha}h_{t+1}^{\zeta}-x_{1t+1}-nx_{t+1})^{\xi_{cc}}(1-n_{1t+1}-e_{1t+1})^{\xi_{cl}}\zeta A_{1t+1}v_{1t+1}^{s_{ak}}k_{1t+1}^{s_{ak}}n_{1t+1}^{s_{ah}}h_{t+1}^{\zeta-1}+\ldots\\
&\ldots+\tilde{\mu}_{1t+1}(1-\delta_H)+\chi\tilde{\mu}_{1t+1}B_{1t+1}(1-v_{1t+1})^{s_{bk}}k_{1t+1}^{s_{bk}}e_{1t+1}^{s_{bh}}h_{t+1}^{\chi-1}-\frac{\tilde{\mu}_{1t}\gamma_t^{-\xi_{cc}}}{\beta}=0\ ,
\end{aligned}
\tag{37}
$$

$$
\begin{aligned}
&\Pi_2\tau[A_{2t+1}v_{2t+1}^{s_{ak}}k_{2t+1}^{s_{ak}}n_{2t+1}^{\alpha}(1-h_{t+1})^{\zeta}-x_{2t+1}+\pi nx_{t+1}]^{\xi_{cc}}(1-n_{2t+1}-e_{2t+1})^{\xi_{cl}}\zeta A_{2t+1}v_{2t+1}^{s_{ak}}k_{2t+1}^{s_{ak}}n_{2t+1}^{s_{ah}}(1-h_{t+1}\\
&\ldots+\tilde{\mu}_{2t+1}(1-\delta_H)+\chi\tilde{\mu}_{2t+1}B_{2t+1}(1-v_{2t+1})^{s_{bk}}k_{2t+1}^{s_{bk}}e_{2t+1}^{s_{bh}}(1-h_{t+1})^{\chi-1}-\frac{\tilde{\mu}_{2t}\gamma_t^{-\xi_{cc}}}{\beta}=0\ ,
\end{aligned}
\tag{38}
$$

g) w.r.t. λ_{it}:

$$(1-\delta_K)k_{it} + \psi\left(\frac{x_{it}}{k_{it}}\right)k_{it} - \gamma_t k_{it+1} = 0 \ . \tag{39}$$

h) w.r.t. μ_{it}:

$$(1-\delta_H)h_t + B_{1t}(1-v_{1t})^{s_{bk}} k_{1t}^{s_{bk}} e_{1t}^{s_{bh}} h_t^{\chi} - \gamma_t h_{t+1} = 0 \ . \tag{40}$$

$$(1-\delta_H)(1-h_t) + B_{2t}(1-v_{2t})^{s_{bk}} k_{2t}^{s_{bk}} e_{1t}^{s_{bh}} (1-h_t)^{\chi} - \gamma_t(1-h_{t+1}) = 0 \ . \tag{41}$$

Substituting (39) into (40) we are able to eliminate h_{t+1} and get:

$$(1-\delta_H) + B_{1t}(1-v_{1t})^{s_{bk}} k_{1t}^{s_{bk}} e_{1t}^{s_{bh}} h_t^{\chi} + B_{2t}(1-v_{2t})^{s_{bk}} k_{2t}^{s_{bk}} e_{2t}^{s_{bh}} (1-h_t)^{\chi} - \gamma_t = 0 \ . \tag{42}$$

5 Steady-state.

In steady state, *nx* has to be zero, because otherwise one of the two countries would continuously accumulate financial wealth. Given *nx*=0, we can easily prove that the steady state allocation is perfectly symmetric across countries. I explicitly calibrate the model to reproduce the empirically observed values for n, e, s_X, s_Z, r_{ky} and γ (where n and e are the long-run time shares devoted to the consumption good sector and the human capital accumulation sector, $s_X \equiv x/y$ is the long-run investments/output ratio, $s_Z \equiv z/y$ is the long-run relative size of the human capital sector vs. the consumption good sector, and $r_{ky} \equiv k/y$ the long-run yearly physical capital/output ratio) by transforming the parameters η, τ, δ_K, δ_H, A and B into endogenous variables. Evaluating the first order conditions at the steady-state we have (taking into account that $h_t = 1 - h_t = h$):

$$\Pi_i(1-\tau)c^{\xi_{lc}} l^{\xi_{ll}} \quad s_{ah}\lambda_i \frac{y}{\omega} = \mu_i s_{bh}\frac{z}{e} \ , \tag{43/44}$$

$$s_{ak}\lambda_i \frac{y}{v} = s_{bk}\mu_i \frac{z}{1-v} \ , \tag{45}$$

$$\Pi_i \tau c^{\xi_{cc}} l^{\xi_{cl}} = \lambda_i \ , \tag{46}$$

$$\lambda_1 = \pi\lambda_2 \ , \tag{47}$$

$$s_{ak}\frac{y}{k} + (1 - \delta_K) + s_{bk}\frac{\mu_i}{\lambda_i}\frac{z}{k} = \frac{\gamma^{-\xi_{cc}}}{\beta} \ , \tag{48}$$

$$\zeta\frac{\lambda_i}{\mu_i}\frac{y}{h} + (1 - \delta_H) + \chi\frac{z}{h} = \frac{\gamma^{-\xi_{cc}}}{\beta} \ , \tag{49}$$

$$\gamma = (1 - \delta_K) + \frac{x}{k} \ , \tag{50}$$

$$\gamma = (1 - \delta_H) + \frac{z}{h} \ . \tag{51}$$

where $\omega \equiv n/l$ and $z \equiv B[(1-v)k]^{s_{bk}} e^{s_{bh}} h^{\chi}$. From (50) we get:

$$\delta_K = \frac{x}{k} + 1 - \gamma \ . \tag{52}$$

Substituting (45) into (48) we get:

$$s_{ak}\frac{y}{k} + (1 - \delta_K) + s_{ak}\frac{y}{k}\frac{1-v}{v} = \frac{\gamma^{-\xi_{cc}}}{\beta} \ , \tag{53}$$

or:

$$v = \frac{\beta s_{ak}}{r_{ky}[\gamma^{-\xi_{cc}} - \beta(1-\delta_K)]} \ . \tag{54}$$

Combining (43), (44) and (46) gets:

$$\frac{\lambda}{\mu} = \frac{s_{bh}}{s_{ah}}\frac{z}{y}\upsilon \ . \tag{55}$$

where $\upsilon \equiv n/e$. Substituting (55) into (45) gets:

$$s_{bh} = \frac{s_{ah}\varphi}{s_{ak}\upsilon + s_{ah}\varphi} , \tag{56}$$

where $\varphi \equiv v/(1-v)$. Substituting now (51) and (55) into (49) we get:

$$\chi(\gamma - 1 + \delta_H)\upsilon + (1 - \delta_H) + \chi(\gamma - 1 + \delta_H) = \frac{\gamma^{-\xi_{cc}}}{\beta} , \tag{57}$$

or:

$$\delta_H = 1 - \frac{\gamma^{-\xi_{cc}} - \gamma\beta\chi(1+\upsilon)}{\beta[1 - \chi(1+\upsilon)]} , \tag{58}$$

From (43) we get:

$$\tau = \frac{s_c\omega}{s_{ah} + s_c\omega} . \tag{59}$$

Consider now the ratio between the human capital sector's output and the consumption good sector's output:

$$\frac{z}{y} = \frac{B(1-v)^{s_{bk}}k^{s_{bk}}e^{s_{bh}}h^{\chi}}{y} , \tag{60}$$

It is easy to show that:

$$y = A^{\frac{1}{s_{ah}}}(vr_{ky})^{\frac{s_{ak}}{s_{ah}}}nh^{\frac{\zeta}{s_{ah}}} , \tag{61}$$

Substituting (51) and (61) into (60), and solving for A gets:

$$A = \frac{[\gamma - (1 - \delta_H)]^{s_{ah}}}{r_{zy}^{s_{ah}}(vr_{ky})^{s_{ak}}n^{s_{ah}}}h^{s_{ah}-\zeta} . \tag{62}$$

Finally, solving (51) for B and taking into account that $k = r_{ky}y$ gets:

$$B = \frac{\gamma - (1 - \delta_H)}{[(1-v)k]^{s_{bk}} e^{s_{bh}} h^{\chi - 1}} \ . \tag{63}$$

6 Log-linearization of the first order conditions.

Given that the system of normalized first order conditions is highly non-linear, and thus non analytically solvable, we have to approximate the system around a stationary point. We do that log-linearizing around the symmetric steady-state determined in the previous section: the log-linearization has the advantage to transform the variables into percentage deviations from their steady-state value. To log-linearize an equation, we substitute to any variable x_{it} the expression $\exp(\log(x_{it}))$ and partially derive with respect to $\log(x_{it})$. The result, evaluated at the steady state, will be the elasticity coefficient that multiples $\hat{x}_{it}$=$\log(x_{it})$-$\log(x)$ in the approximating equation. Of course, since nx_t is equal to zero in steady-state, and its value may be negative, we cannot log-linearize with respect to nx, but we have to linearize in levels: in other words, to nx_t we substitute the expression $(nx_t/y)\,y$, where y is the steady-state value for the normalized output of the consumption good sector, and the derive with respect to nx/y.

a) w.r.t. $\mathbf{n_{it}}$:

$$\tau(1 - n_{1t} - e_{1t}) s_{ah} A_{1t} v_{1t}^{s_{ak}} k_{1t}^{s_{ak}} n_{1t}^{-s_{ak}} h_t^{\zeta} - (1-\tau)(A_{1t} v_{1t}^{s_{ak}} k_{1t}^{s_{ak}} n_{1t}^{s_{ah}} h_t^{\zeta} - x_{1t} - nx_t) = 0 \ , \tag{64}$$

The elasticities are:

$\mathbf{A_{1t}}$**:** $\tau s_{ah}\dfrac{y}{\omega} - (1-\tau)y$; $\mathbf{n_{1t}}$**:** $-\tau s_{ah}\dfrac{y}{\omega}\omega - \tau s_{ak}s_{ah}\dfrac{y}{\omega} - (1-\tau)s_{ah}y$; $\mathbf{e_{1t}}$**:** $-\tau s_{ah}\dfrac{y}{\omega}\varsigma$;

$\mathbf{v_{1t}}$**:** $\tau s_{ak}s_{ah}\dfrac{y}{\omega} - (1-\tau)s_{ak}y$; $\mathbf{x_{1t}}$**:** $(1-\tau)x$; $\mathbf{nx_t}$**:** $(1-\tau)y$; $\mathbf{k_{1t}}$**:** $\tau s_{ak}s_{ah}\dfrac{y}{\omega} - (1-\tau)s_{ak}y$;

$\mathbf{h_t}$**:** $\tau\zeta s_{ah}\dfrac{y}{\omega} - (1-\tau)\zeta y$;

where $\varsigma \equiv e/l$. We can divide everything by $\tau s_{ah}(y/\omega) = (1-\tau)c$, multiply by s_c, and write the log-linearized f.o.c. w.r.t. n_{1t} as:

$$-[s_{ah}+s_c(s_{ak}+\omega)]\hat{n}_{1t}-s_c\varsigma\hat{e}_{1t}-s_{ak}s_x\hat{v}_{1t}+s_x\hat{x}_{1t}+\hat{nx}_t=s_{ak}s_x\hat{k}_{1t}+\zeta s_x\hat{h}_t+s_x\hat{A}_{1t}\ . \tag{65}$$

where $s_x = 1 - s_c$. Symmetrically, the log-linearized f.o.c. w.r.t. n_{2t} can be written as (recall that in steady state $h/(1-h) = 1$):

$$-[s_{ah}+s_c(s_{ak}+\omega)]\hat{n}_{2t}-s_c\varsigma\hat{e}_{2t}-s_{ak}s_x\hat{v}_{2t}+s_x\hat{x}_{2t}-\pi\hat{nx}_t=s_{ak}s_x\hat{k}_{2t}-\zeta s_x\hat{h}_t+s_x\hat{A}_{2t}\ . \tag{66}$$

b) w.r.t. e_{1t}:

$$-\Pi_1(1-\tau)(A_{1t}v_{1t}^{s_{ak}}k_{1t}^{s_{ak}}n_{1t}^{s_{ah}}h_t^{\zeta}-x_{1t}-nx_t)^{\xi_{lc}}(1-n_{1t}-e_{1t})^{\xi_{ll}}+s_{bh}\mu_{1t}B_{1t}(1-v_{1t})^{s_{bk}}k_{1t}^{s_{bk}}e_{1t}^{-s_{bk}}h_t^{\chi}=0\ . \tag{67}$$

A_{1t}: $-\xi_{lc}\Pi_1(1-\tau)c^{\xi_{lc}-1}l^{\xi_{ll}}y$; **$B_{1t}$:** $s_{bh}\mu_1\frac{z}{e}$; **n_{1t}:** $-\xi_{lc}\Pi_1(1-\tau)c^{\xi_{lc}-1}l^{\xi_{ll}}s_{ah}y+\xi_{ll}\Pi_1(1-\tau)c^{\xi_{lc}}l^{\xi_{ll}}\omega$;

e_{1t}: $\xi_{ll}\Pi_1(1-\tau)c^{\xi_{lc}}l^{\xi_{ll}}\varsigma-s_{bk}s_{bh}\mu_1\frac{z}{e}$; **$v_{1t}$:** $-\xi_{lc}\Pi_1(1-\tau)c^{\xi_{lc}-1}l^{\xi_{ll}}s_{ak}y-s_{bk}s_{bh}\mu_1\frac{z}{e}\varphi$;

x_{1t}: $\xi_{lc}\Pi_1(1-\tau)c^{\xi_{lc}-1}l^{\xi_{ll}}x$; **$nx_t$:** $\xi_{lc}\Pi_1(1-\tau)c^{\xi_{lc}}l^{\xi_{ll}}\frac{y}{s_c}$; **$k_{1t}$:** $-\xi_{lc}\Pi_1(1-\tau)c^{\xi_{lc}-1}l^{\xi_{ll}}s_{ak}y+s_{bk}s_{bh}\mu_1\frac{z}{e}$;

h_t: $-\xi_{lc}\Pi_1(1-\tau)c^{\xi_{lc}-1}l^{\xi_{ll}}\zeta y+\chi s_{bh}\mu_1\frac{z}{e}$; **$\mu_{1t}$:** $s_{bh}\mu_1\frac{z}{e}$

We can divide everything by $\Pi_1(1-\tau)c^{\xi_{lc}}l^{\xi_{ll}}=s_{bh}\mu_1(z/e)$ and multiplying by s_c, writing the log-linearized first order condition w.r.t. e_{1t} as:

$$\begin{aligned}&(s_c\xi_{ll}\omega-\xi_{lc}s_{ah})\hat{n}_{1t}+s_c(\xi_{ll}\varsigma-s_{bk})\hat{e}_{1t}-(\xi_{lc}s_{ak}+s_cs_{bk}\varphi)\hat{v}_{1t}+\xi_{lc}s_x\hat{x}_{1t}+\xi_{lc}\hat{nx}_t=\\&=(\xi_{lc}s_{ak}-s_cs_{bk})\hat{k}_{1t}+(\xi_{lc}\zeta-s_c\chi)\hat{h}_t-s_c\hat{\mu}_{1t}+\xi_{lc}\hat{A}_{1t}-s_c\hat{B}_{1t}\ .\end{aligned} \tag{68}$$

Symmetrically, the log-linearized first order condition w.r.t. e_{2t} can be written as:

$$\boxed{\begin{aligned}&(s_c\xi_{ll}\omega-\xi_{lc}s_{ah})\hat{n}_{2t}+s_c(\xi_{ll}\varsigma-s_{bk})\hat{e}_{2t}-(\xi_{lc}s_{ak}+s_c s_{bk}\varphi)\hat{v}_{2t}+\xi_{lc}s_x\hat{x}_{2t}-\xi_{lc}\pi\hat{nx}_t=\\&\quad=(\xi_{lc}s_{ak}-s_c s_{bk})\hat{k}_{2t}-(\xi_{lc}\zeta-s_c\chi)\hat{h}_t-s_c\hat{\mu}_{2t}+\xi_{lc}\hat{A}_{2t}-s_c\hat{B}_{2t}\ .\end{aligned}} \tag{69}$$

c) w.r.t. v_{1t}:

$$\begin{aligned}&\Pi_1\tau(A_{1t}v_{1t}^{s_{ak}}k_{1t}^{s_{ak}}n_{1t}^{s_{ah}}h_t^{\zeta}-x_{1t}-nx_{1t})^{\xi_{cc}}(1-n_{1t}-e_{1t})^{\xi_{cl}}s_{ak}A_{1t}v_{1t}^{-s_{ah}}k_{1t}^{s_{ak}}n_{1t}^{s_{ah}}h_t^{\zeta}+\ldots\\&\qquad\ldots-s_{bk}\mu_{1t}B_{1t}(1-v_{1t})^{-s_{bh}}k_{1t}^{s_{bk}}e_{1t}^{s_{bh}}h_t^{\chi}=0\ .\end{aligned} \tag{70}$$

$\mathbf{A_{1t}}$: $\xi_{cc}\lambda_1 s_{ak}\frac{y}{v}\frac{1}{s_c}+\lambda_1 s_{ak}\frac{y}{v}$; $\mathbf{B_{1t}}$: $-s_{bk}\mu_1\frac{z}{1-v}$; $\mathbf{n_{1t}}$: $\xi_{cc}\lambda_1 s_{ak}\frac{y}{v}\frac{s_{ah}}{s_c}-\xi_{cl}\lambda_1 s_{ak}\frac{y}{v}\omega+s_{ah}\lambda_1 s_{ak}\frac{y}{v}$;

$\mathbf{e_{1t}}$: $-\xi_{cl}\lambda_1 s_{ak}\frac{y}{v}\varsigma-s_{bh}s_{bk}\mu_1\frac{z}{1-v}$; $\mathbf{v_{1t}}$: $\xi_{cc}\lambda_1 s_{ak}^2\frac{y}{v}\frac{1}{s_c}-\lambda_1 s_{ak}\frac{y}{v}s_{ah}-s_{bh}s_{bk}\mu_1\frac{z}{1-v}\varphi$; $\mathbf{x_{1t}}$: $-\xi_{cc}\lambda_1 s_{ak}\frac{y}{v}\frac{s_x}{s_c}$;

$\mathbf{nx_t}$: $-\xi_{cc}\lambda_1 s_{ak}\frac{y}{v}\frac{1}{s_c}$; $\mathbf{k_{1t}}$: $\xi_{cc}\lambda_1 s_{ak}^2\frac{y}{v}\frac{1}{s_c}+\lambda_1 s_{ak}^2\frac{y}{v}-s_{bk}^2\mu_1\frac{z}{1-v}$; $\mathbf{h_t}$: $\xi_{cc}\lambda_1\zeta s_{ak}\frac{y}{v}\frac{1}{s_c}+\lambda_1\zeta\frac{y}{v}-\chi s_{bk}\mu_1\frac{z}{1-v}$;

$\mathbf{\mu_{1t}}$: $-s_{bk}\mu_1\frac{z}{1-v}$;

We can divide everything by:

$$\lambda_1 s_{ak}\frac{y}{v}=s_{bk}\mu_1\frac{z}{1-v}\ , \tag{71}$$

Multiplying everything by s_c, we may write the log-linearized first order condition w.r.t. v_{1t} as:

$$\boxed{\begin{aligned}&[\xi_{cc}s_{ah}+s_c(s_{ah}-\xi_{cl}\omega)]\hat{n}_{1t}-s_c(\xi_{cl}\varsigma+s_{bh})\hat{e}_{1t}+[\xi_{cc}s_{ak}-s_c(s_{ah}+s_{bh}\varphi)]\hat{v}_{1t}-\xi_{cc}s_x\hat{x}_{1t}-\xi_{cc}\hat{nx}_t=\\&\quad=[s_c(s_{bk}-s_{ak})-\xi_{cc}s_{ak}]\hat{k}_{1t}+[s_c(\chi-\zeta)-\xi_{cc}\zeta]\hat{h}_t+s_c\hat{\mu}_{1t}-(\xi_{cc}+s_c)\hat{A}_{1t}+s_c\hat{B}_{1t}\ .\end{aligned}} \tag{72}$$

Symmetrically, the log-linearized first order condition w.r.t. v_{2t} can be written as:

$$\boxed{\begin{aligned}&[\xi_{cc}s_{ah}+s_c(s_{ah}-\xi_{cl}\omega)]\hat{n}_{2t}-s_c(\xi_{cl}\varsigma+s_{bh})\hat{e}_{2t}+[\xi_{cc}s_{ak}-s_c(s_{ah}+s_{bh}\varphi)]\hat{v}_{2t}-\xi_{cc}s_x\hat{x}_{2t}+\xi_{cc}\pi\hat{nx}_t=\\&=[s_c(s_{bk}-s_{ak})-\xi_{cc}s_{ak}]\hat{k}_{2t}-[s_c(\chi-\zeta)-\xi_{cc}\zeta]\hat{h}_t+s_c\hat{\mu}_{2t}-(\xi_{cc}+s_c)\hat{A}_{2t}+s_c\hat{B}_{2t}\ .\end{aligned}} \tag{73}$$

d) w.r.t. X_{1t}:

$$\psi'\left(\frac{x_{1t}}{k_{1t}}\right)\lambda_{1t}-\Pi_1\tau(A_{1t}v_{1t}^{s_{ak}}k_{1t}^{s_{ak}}n_{1t}^{s_{ah}}h_t^{\zeta}-x_{1t}-nx_t)^{\xi_{cc}}(1-n_{1t}-e_{1t})^{\xi_{cl}}=0\ . \tag{74}$$

A_{1t}: $-\xi_{cc}\lambda_1\frac{1}{s_c}$; **$n_{1t}$:** $-\xi_{cc}\lambda_1\frac{s_{ah}}{s_c}+\xi_{cl}\lambda_1\omega$; **$e_{1t}$:** $\xi_{cl}\lambda_1\varsigma$; **$v_{1t}$:** $-\xi_{cc}\lambda_1\frac{s_{ak}}{s_c}$; **$x_{1t}$:** $\xi_{cc}\lambda_1\frac{s_x}{s_c}-\xi_\psi\lambda_1$;

nx_t: $\xi_{cc}\lambda_1\frac{1}{s_c}$; **$k_{1t}$:** $-\xi_{cc}\lambda_1\frac{s_{ak}}{s_c}+\xi_\psi\lambda_1$; **$h_t$:** $-\xi_{cc}\lambda_1\frac{\zeta}{s_c}$; **$\lambda_{1t}$:** λ_1.

Now, dividing everything by λ_1 and multiplying by s_c, we may write the log-linearized first order condition w.r.t. x_{1t} as:

$$\boxed{\begin{aligned}&(s_c\xi_{cl}\omega-\xi_{cc}s_{ah})\hat{n}_{1t}+s_c\xi_{cl}\varsigma\hat{e}_{1t}-\xi_{cc}s_{ak}\hat{v}_{1t}+(\xi_{cc}s_x-s_c\xi_\psi)\hat{x}_{1t}+\xi_{cc}\hat{nx}_t=\\&=(\xi_{cc}s_{ak}-s_c\xi_\psi)\hat{k}_{1t}+\xi_{cc}\zeta\hat{h}_t-s_c\hat{\lambda}_{1t}+\xi_{cc}\hat{A}_{1t}\ .\end{aligned}} \tag{75}$$

Symmetrically, the log-linearized first order condition w.r.t. x_{2t} can be written as**:**

$$\boxed{\begin{aligned}&(s_c\xi_{cl}\omega-\xi_{cc}s_{ah})\hat{n}_{2t}+s_c\xi_{cl}\varsigma\hat{e}_{2t}-\xi_{cc}s_{ak}\hat{v}_{2t}+(\xi_{cc}s_x-s_c\xi_\psi)\hat{x}_{2t}-\xi_{cc}\pi\hat{nx}_t=\\&=(\xi_{cc}s_{ak}-s_c\xi_\psi)\hat{k}_{2t}-\xi_{cc}\zeta\hat{h}_t-s_c\hat{\lambda}_{2t}+\xi_{cc}\hat{A}_{2t}\ .\end{aligned}} \tag{76}$$

e) First order condition w.r.t. NX_t:

$$\begin{aligned}&-\tau(A_{1t}v_{1t}^{s_{ak}}k_{1t}^{s_{ak}}n_{1t}^{s_{ah}}h_t^{\zeta}-x_{1t}-nx_t)^{\xi_{cc}}(1-n_{1t}-e_{1t})^{\xi_{cl}}+\ldots\\&\ldots+\tau[A_{2t}v_{2t}^{s_{ak}}k_{2t}^{s_{ak}}n_{2t}^{s_{ah}}(1-h_t)^{\zeta}-x_{2t}+\pi nx_t]^{\xi_{cc}}(1-n_{2t}-e_{2t})^{\xi_{cl}}=0\ .\end{aligned} \tag{77}$$

$\mathbf{A_{1t}}$: $-\xi_{cc}\tau c^{\xi_{cc}-1}l^{\xi_{cl}}y$; $\mathbf{A_{2t}}$: $\xi_{cc}\tau c^{\xi_{cc}-1}l^{\xi_{cl}}y$; $\mathbf{n_{1t}}$: $-\xi_{cc}\tau c^{\xi_{cc}-1}l^{\xi_{cl}}s_{ah}y+\xi_{cl}\tau c^{\xi_{cc}}l^{\xi_{cl}}\omega$;

$\mathbf{n_{2t}}$: $\xi_{cc}\tau c^{\xi_{cc}-1}l^{\xi_{cl}}s_{ah}y-\xi_{cl}\tau c^{\xi_{cc}}l^{\xi_{cl}}\omega$; $\mathbf{e_{1t}}$: $\xi_{cl}\tau c^{\xi_{cc}}l^{\xi_{cl}}\varsigma$; $\mathbf{e_{2t}}$: $-\xi_{cl}\tau c^{\xi_{cc}}l^{\xi_{cl}}\varsigma$;

$\mathbf{v_{1t}}$: $-\xi_{cc}\tau c^{\xi_{cc}-1}l^{\xi_{cl}}s_{ak}y$; $\mathbf{v_{2t}}$: $\xi_{cc}\tau c^{\xi_{cc}-1}l^{\xi_{cl}}s_{ak}y$; $\mathbf{x_{1t}}$: $\xi_{cc}\tau c^{\xi_{cc}-1}l^{\xi_{cl}}x$; $\mathbf{x_{2t}}$: $-\xi_{cc}\tau c^{\xi_{cc}-1}l^{\xi_{cl}}x$

$\mathbf{nx_t}$: $\xi_{cc}\tau c^{\xi_{cc}}l^{\xi_{cl}}\frac{1}{\Pi_2}\frac{1}{s_c}$; $\mathbf{k_{1t}}$: $-\xi_{cc}\tau c^{\xi_{cc}-1}l^{\xi_{cl}}s_{ak}y$; $\mathbf{k_{2t}}$: $\xi_{cc}\tau c^{\xi_{cc}-1}l^{\xi_{cl}}s_{ak}y$; $\mathbf{h_t}$: 0;

Dividing everything by $\tau c^{\xi_{cc}}l^{\xi_{cl}}$ and multiplying by s_c, we may write the log-linearized first order condition w.r.t. nx_t as:

$$\boxed{\begin{aligned}&(s_c\xi_{cl}\omega-\xi_{cc}s_{ah})\hat{n}_{1t}-(s_c\xi_{cl}\omega-\xi_{cc}s_{ah})\hat{n}_{2t}+s_c\xi_{cl}\varsigma\hat{e}_{1t}-s_c\xi_{cl}\varsigma\hat{e}_{2t}-\xi_{cc}s_{ak}\hat{v}_{1t}+\xi_{cc}s_{ak}\hat{v}_{2t}+\dots\\&\dots+\xi_{cc}s_x\hat{x}_{1t}-\xi_{cc}s_x\hat{x}_{2t}+\frac{\xi_{cc}}{\Pi_2}\hat{nx}_t=\xi_{cc}s_{ak}\hat{k}_{1t}-\xi_{cc}s_{ak}\hat{k}_{2t}+\xi_{cc}\hat{A}_{1t}-\xi_{cc}\hat{A}_{2t}\ .\end{aligned}} \tag{78}$$

f) First order condition w.r.t. μ_{2t}:[1]

$$(1-\delta_H)+B_{1t}(1-v_{1t})^{1-\eta}k_{1t}^{1-\eta}e_{1t}^{\eta}h_t^{\chi}+B_{2t}(1-v_{2t})^{1-\eta}k_{2t}^{1-\eta}e_{2t}^{\eta}(1-h_t)^{\chi}-\gamma_t=0\ . \tag{79}$$

$\boldsymbol{\gamma_t}$: $-\gamma$; $\mathbf{B_{it}}$: z; $\mathbf{e_{it}}$: $s_{bh}z$; $\mathbf{v_{it}}$: $-s_{bk}z\varphi$; $\mathbf{k_{it}}$: $s_{bk}z$; $\mathbf{h_t}$: 0.

Now, dividing everything by z, we can write the log-linearized expression for γ_t as:

$$\boxed{s_{bh}\hat{e}_{1t}+s_{bh}\hat{e}_{2t}-s_{bk}\varphi\hat{v}_{1t}-s_{bk}\varphi\hat{v}_{2t}-\frac{\gamma}{z}\hat{\gamma}_{2t}=-s_{bk}\hat{k}_{1t}-s_{bk}\hat{k}_{2t}-\hat{B}_{1t}-\hat{B}_{2t}\ .} \tag{80}$$

g) First order condition w.r.t. K_{1t+1}:

[1] This equation describes the behaviour of γ_t as a function of the contemporaneous control, control-like, and state-like variables: that's why γ_t can be considered as a further control-like variable.

$$\Pi_1 \tau (A_{1t+1} v_{1t+1}^{s_{ak}} k_{1t+1}^{s_{ak}} n_{1t+1}^{s_{ah}} h_{t+1}^{\zeta} - x_{1t+1} - nx_{t+1})^{\xi_{cc}} (1 - n_{1t+1} - e_{1t+1})^{\xi_{cl}} s_{ak} A_{1t+1} v_{1t+1}^{s_{ak}} k_{1t+1}^{-s_{ah}} n_{1t+1}^{s_{ah}} h_{t+1}^{\zeta} + \ldots$$

$$\ldots + \tilde{\lambda}_{1t+1}\left[(1-\delta_K) + \psi\left(\frac{x_{1t+1}}{k_{1t+1}}\right) - \psi'\left(\frac{x_{1t+1}}{k_{1t+1}}\right)\frac{x_{1t+1}}{k_{1t+1}}\right] + s_{bk}\tilde{\mu}_{1t+1} B_{1t+1}(1-v_{1t+1})^{s_{bk}} k_{1t+1}^{-s_{bh}} e_{1t+1}^{s_{bh}} h_{t+1}^{\chi} - \frac{\tilde{\lambda}_{1t}\gamma_t^{-\xi_{cc}}}{\beta} = (\qquad (81)$$

$\mathbf{A_{1t+1}}$: $\xi_{cc}\lambda s_{ak}\frac{y}{k}\frac{1}{s_c} + \Pi_1 \lambda s_{ak}\frac{y}{k}$; $\mathbf{B_{1t+1}}$: $s_{bk}\mu_1\frac{z}{k}$; $\mathbf{n_{1t+1}}$: $\xi_{cc}\lambda s_{ak}\frac{y}{k}\frac{s_{ah}}{s_c} - \xi_{cl}\lambda s_{ak}\frac{y}{k}\omega + \lambda s_{ak}\frac{y}{k}s_{ah}$;

$\mathbf{e_{1t+1}}$: $-\xi_{cl}\lambda s_{ak}\frac{y}{k}\varsigma + s_{bh}s_{bk}\mu_1\frac{z}{k}$; $\mathbf{v_{1t+1}}$: $\xi_{cc}\lambda\frac{y}{k}\frac{s_{ak}^2}{s_c} + \lambda\frac{y}{k}s_{ak}^2 - s_{bk}^2\mu_1\frac{z}{k}\varphi$; $\mathbf{x_{1t+1}}$: $-\xi_{cc}\lambda s_{ak}\frac{y}{k}\frac{s_x}{s_c} + \lambda_1\xi_\psi r_{xk}$;

$\mathbf{nx_{t+1}}$: $-\xi_{cc}\lambda s_{ak}\frac{y}{k}\frac{1}{s_c}$; $\mathbf{k_{1t+1}}$: $\xi_{cc}\lambda s_{ak}^2\frac{y}{k}\frac{1}{s_c} - \lambda s_{ak}\frac{y}{k}s_{ah} - \lambda_1\xi_\psi r_{xk} - s_{bh}s_{bk}\mu_1\frac{z}{k}$;

$\mathbf{h_{t+1}}$: $\xi_{cc}\lambda\zeta s_{ak}\frac{y}{k}\frac{1}{s_c} + \lambda\zeta s_{ak}\frac{y}{k} + \chi s_{bk}\mu_1\frac{z}{k}$; λ_{1t}: $-\frac{\lambda_1\gamma^{-\xi_{cc}}}{\beta}$; λ_{1t+1}: $\lambda_1(1-\delta_K)$;

μ_{1t+1}: $s_{bk}\mu_1\frac{z}{k}$; γ_t: $\xi_{cc}\frac{\lambda_1\gamma^{-\xi_{cc}}}{\beta}$.

We may divide everything by $(\lambda_1 s_{ak})/r_{ky}$, taking into account that, from (42):

$$s_{bk}\mu_1\frac{z}{k} = \frac{s_{ak}\lambda_1}{\varphi r_{ky}} \ . \qquad (82)$$

and get:

$\mathbf{A_{1t+1}}$: $\frac{\xi_{cc}}{s_c} + 1$; $\mathbf{B_{1t+1}}$: $\frac{1}{\varphi}$; $\mathbf{n_{1t+1}}$: $\frac{\xi_{cc}s_{ah}}{s_c} - \xi_{cl}\omega + s_{ah}$; $\mathbf{e_{1t+1}}$: $-\xi_{cl}\varsigma + \frac{s_{bh}}{\varphi}$; $\mathbf{v_{1t+1}}$: $\frac{\xi_{cc}s_{ak}}{s_c} + s_{ak} - s_{bk}$;

$\mathbf{x_{1t+1}}$: $-\frac{\xi_{cc}s_x}{s_c} + \xi_\psi r_{xk}\frac{r_{ky}}{s_{ak}}$; $\mathbf{nx_{t+1}}$: $-\frac{\xi_{cc}}{s_c}$; $\mathbf{k_{1t+1}}$: $\frac{\xi_{cc}s_{ak}}{s_c} - s_{ah} - \xi_\psi r_{xk}\frac{r_{ky}}{s_{ak}} - \frac{s_{bh}}{\varphi}$; $\mathbf{h_{t+1}}$: $\frac{\xi_{cc}\zeta}{s_c} + \zeta + \frac{\chi}{\varphi}$;

λ_{1t}: $-\frac{\gamma^{-\xi_{cc}}}{\beta}\frac{r_{ky}}{s_{ak}}$; λ_{1t+1}: $(1-\delta_K)\frac{r_{ky}}{s_{ak}}$; μ_{1t+1}: $\frac{1}{\varphi}$; γ_t: $\xi_{cc}\frac{\gamma^{-\xi_{cc}}}{\beta}\frac{r_{ky}}{s_{ak}}$.

Multiplying everything by s_c, we may write the log-linearized first order condition w.r.t. k_{1t+1} as:

$$\boxed{\begin{aligned}&[\xi_{cc}s_{ak} - s_c(s_{ah} + \frac{s_{bh}}{\varphi}) - \xi_\psi r_{xk}\Delta]\hat{k}_{1t+1} + [\xi_{cc}\zeta + s_c(\zeta + \frac{\chi}{\varphi})]\hat{h}_{t+1} + (1-\delta_K)\Delta\hat{\lambda}_{1t+1} + \frac{s_c}{\varphi}\hat{\mu}_{1t+1} - \frac{\gamma^{-\xi_{cc}}}{\beta}\Delta\hat{\lambda}_{1t} \\ &= [s_c(\xi_{cl}\omega - s_{ah}) - \xi_{cc}s_{ah}]\hat{n}_{1t+1} + s_c(\xi_{cl}\varsigma - \frac{s_{bh}}{\varphi})\hat{e}_{1t+1} + [s_c(s_{bk} - s_{ak}) - \xi_{cc}s_{ak}]\hat{v}_{1t+1} + (\xi_{cc}s_x - \xi_\psi r_{xk}\Delta)\hat{x}_{1t+1} \\ &\quad ... + \xi_{cc}n\hat{x}_{t+1} - \xi_{cc}\frac{\gamma^{-\xi_{cc}}}{\beta}\Delta\hat{\gamma}_t - (\xi_{cc} + s_c)\hat{A}_{1t+1} - \frac{s_c}{\varphi}\hat{B}_{1t+1}\ .\end{aligned}} \quad (83)$$

where $\Delta \equiv (s_c r_{ky})/s_{ak}$. Symmetrically, the log-linearized first order condition w.r.t. K_{2t+1} can be written as:

$$\boxed{\begin{aligned}&[\xi_{cc}s_{ak} - s_c(s_{ah} + \frac{s_{bh}}{\varphi}) - \xi_\psi r_{xk}\Delta]\hat{k}_{2t+1} - [\xi_{cc}\zeta + s_c(\zeta + \frac{\chi}{\varphi})]\hat{h}_{t+1} + (1-\delta_K)\Delta\hat{\lambda}_{2t+1} + \frac{s_c}{\varphi}\hat{\mu}_{2t+1} - \frac{\gamma^{-\xi_{cc}}}{\beta}\Delta\hat{\lambda}_{2t} \\ &= [s_c(\xi_{cl}\omega - s_{ah}) - \xi_{cc}s_{ah}]\hat{n}_{2t+1} + s_c(\xi_{cl}\varsigma - \frac{s_{bh}}{\varphi})\hat{e}_{2t+1} + [s_c(s_{bk} - s_{ak}) - \xi_{cc}s_{ak}]\hat{v}_{2t+1} + (\xi_{cc}s_x - \xi_\psi r_{xk}\Delta)\hat{x}_{2t+1} \\ &\quad ... - \xi_{cc}\pi n\hat{x}_{t+1} - \xi_{cc}\frac{\gamma^{-\xi_{cc}}}{\beta}\Delta\hat{\gamma}_t - (\xi_{cc} + s_c)\hat{A}_{2t+1} - \frac{s_c}{\varphi}\hat{B}_{2t+1}\ .\end{aligned}} \quad (84)$$

h) First order conditions w.r.t. $\mathbf{H_{1t+1}}$:

$$\begin{aligned}&\Pi_1\tau(A_{1t+1}v_{1t+1}^{s_{ak}}k_{1t+1}^{s_{ak}}n_{1t+1}^{s_{ah}}h_{t+1}^{\zeta} - x_{1t+1} - nx_{1t+1})^{\xi_{cc}}(1 - n_{1t+1} - e_{1t+1})^{\xi_{cl}}\zeta A_{1t+1}v_{1t+1}^{s_{ak}}k_{1t+1}^{s_{ak}}n_{1t+1}^{s_{ah}}h_{t+1}^{\zeta-1} + \mu_{1t+1}(1 \\ &\quad ... + \chi\mu_{1t+1}B_{1t+1}(1 - v_{1t+1})^{s_{bk}}k_{1t+1}^{s_{bk}}e_{1t+1}^{s_{bh}}h_{t+1}^{\chi-1} - \frac{\mu_{1t}\gamma_t^{-\xi_{cc}}}{\beta} = 0\ .\end{aligned} \quad (85)$$

$\mathbf{A_{1t+1}}$: $\xi_{cc}\lambda_1\zeta\frac{y}{h}\frac{1}{s_c} + \lambda_1\zeta\frac{y}{h}$; $\mathbf{B_{1t+1}}$: $\chi\mu_1\frac{z}{h}$; $\mathbf{n_{1t+1}}$: $\xi_{cc}\lambda_1\zeta\frac{y}{h}\frac{s_{ah}}{s_c} - \xi_{cl}\lambda_1\zeta\frac{y}{h}\omega + \lambda_1\zeta\frac{y}{h}s_{ah}$;

$\mathbf{e_{1t+1}}$: $-\xi_{cl}\lambda_1\zeta\frac{y}{h}\varsigma + s_{bh}\chi\mu_1\frac{z}{h}$; $\mathbf{v_{1t+1}}$: $\xi_{cc}\lambda_1\zeta\frac{y}{h}\frac{s_{ak}}{s_c} + \lambda_1\zeta\frac{y}{h}s_{ak} - s_{bk}\chi\mu_1\frac{z}{h}\varphi$; $\mathbf{x_{1t+1}}$: $-\xi_{cc}\lambda_1\zeta\frac{y}{h}\frac{s_x}{s_c}$;

$\mathbf{nx_{t+1}}$: $-\xi_{cc}\lambda_1\zeta\frac{y}{h}\frac{1}{s_c}$; $\mathbf{k_{1t+1}}$: $\xi_{cc}\lambda_1\zeta\frac{y}{h}\frac{s_{ak}}{s_c} + \lambda_1\zeta\frac{y}{h}s_{ak} + s_{bk}\chi\mu_1\frac{z}{h}$;

$\mathbf{h}_{t+1}$**:** $\xi_{cc}\lambda_1\zeta^2\frac{y}{h}\frac{1}{s_c} - \lambda_1\zeta\frac{y}{h}(1-\zeta) - (1-\chi)\chi\mu_1\frac{z}{h}$; μ_{1t}**:** $-\frac{\mu_1\gamma^{-\xi_{cc}}}{\beta}$; μ_{1t+1}**:** $\mu_1(1-\delta_H) + \chi\mu_1\frac{z}{h}$; γ_t**:** $\xi_{cc}\frac{\mu_1\gamma^{-\xi_{cc}}}{\beta}$.

Now, dividing everything by $\lambda_1\zeta(y/h)$ and taking into account that:

$$\chi\mu_1\frac{z}{h} = \zeta\lambda_1\frac{y}{h}\frac{1}{\upsilon}\ , \tag{86}$$

we get:

$\mathbf{A}_{1t+1}$**:** $\frac{\xi_{cc}}{s_c} + 1$; $\mathbf{B}_{1t+1}$**:** $\frac{1}{\upsilon}$; $\mathbf{n}_{1t+1}$**:** $\frac{\xi_{cc}s_{ah}}{s_c} - \xi_{cl}\omega + s_{ah}$; $\mathbf{e}_{1t+1}$**:** $-\xi_{cl}\varsigma + \frac{s_{bh}}{\upsilon}$;

$\mathbf{v}_{1t+1}$**:** $\frac{\xi_{cc}s_{ak}}{s_c} + s_{ak} - \frac{s_{bk}\varphi}{\upsilon}$; $\mathbf{x}_{1t+1}$**:** $-\frac{\xi_{cc}s_x}{s_c}$; $\mathbf{nx}_{t+1}$**:** $-\frac{\xi_{cc}}{s_c}$; $\mathbf{k}_{1t+1}$**:** $\frac{\xi_{cc}s_{ak}}{s_c} + s_{ak} + \frac{s_{bk}}{\upsilon}$;

$\mathbf{h}_{t+1}$**:** $\frac{\xi_{cc}\zeta}{s_c} - (1-\zeta) - \frac{1-\chi}{\upsilon}$; μ_{1t}**:** $-\frac{\gamma^{-\xi_{cc}}}{\beta\chi[\gamma-(1-\delta_H)]\upsilon}$; μ_{1t+1}**:** $\frac{1-\delta_H}{\chi[\gamma-(1-\delta_H)]\upsilon} + \frac{1}{\upsilon}$;

γ_t**:** $\xi_{cc}\frac{\gamma^{-\xi_{cc}}}{\beta\chi[\gamma-(1-\delta_H)]\upsilon}$.

Multiplying everything by s_c, we may write the linearized first order condition w.r.t. h_{1t+1} as:

$$\boxed{\begin{gathered}[\xi_{cc}s_{ak} + s_c(s_{ak} + \frac{s_{bk}}{\upsilon})]\hat{k}_{1t+1} + [\xi_{cc}\zeta - s_c(1-\zeta+\frac{1-\chi}{\upsilon})]\hat{h}_{t+1} + [(1-\delta_H)\Theta + \frac{s_c}{\upsilon}]\hat{\mu}_{1t+1} - \frac{\gamma^{-\xi_{cc}}}{\beta}\Theta\hat{\mu}_{1t} = \\ = [s_c(\xi_{cl}\omega - s_{ah}) - \xi_{cc}s_{ah}]\hat{n}_{1t+1} + s_c(\xi_{cl}\varsigma - \frac{s_{bh}}{\upsilon})\hat{e}_{1t+1} + [s_c(s_{bk}\frac{\varphi}{\upsilon} - s_{ak}) - \xi_{cc}s_{ak}]\hat{v}_{1t+1} + \xi_{cc}s_x\hat{x}_{1t+1} + \xi_{cc}\hat{nx}_t \\ \ldots - \xi_{cc}\frac{\gamma^{-\xi_{cc}}}{\beta}\Theta\hat{\gamma}_t - (\xi_{cc}+s_c)\hat{A}_{1t+1} - \frac{s_c}{\upsilon}\hat{B}_{1t+1}\ .\end{gathered}} \tag{87}$$

where $\Theta \equiv s_c/(\chi z\upsilon)$. Symmetrically, the log-linearized first order condition w.r.t. h_{2t+1} can be written as:

$$
\begin{aligned}
&[\xi_{cc}s_{ak}+s_c(s_{ak}+\frac{s_{bk}}{\upsilon})]\hat{k}_{2t+1}-[\xi_{cc}\zeta-s_c(1-\zeta+\frac{1-\chi}{\upsilon})]\hat{h}_{t+1}+[(1-\delta_H)\Theta+\frac{s_c}{\upsilon}]\hat{\mu}_{2t+1}-\frac{\gamma^{-\xi_{cc}}}{\beta}\Theta\hat{\mu}_{2t}= \\
&=[s_c(\xi_{cl}\omega-s_{ah})-\xi_{cc}s_{ah}]\hat{n}_{2t+1}+s_c(\xi_{cl}\varsigma-\frac{s_{bh}}{\upsilon})\hat{e}_{2t+1}+[s_c(s_{bk}\frac{\varphi}{\upsilon}-s_{ak})-\xi_{cc}s_{ak}]\hat{v}_{2t+1}+\xi_{cc}s_x\hat{x}_{2t+1}-\xi_{cc}\pi\hat{n} \\
&\ldots-\xi_{cc}\frac{\gamma^{-\xi_{cc}}}{\beta}\Theta\hat{\gamma}_t-(\xi_{cc}+s_c)\hat{A}_{2t+1}-\frac{s_c}{\upsilon}\hat{B}_{2t+1}\ .
\end{aligned}
\tag{88}
$$

i) First order conditions w.r.t. λ_{it}:

$$(1-\delta_K)k_{it}+\psi(\frac{x_{it}}{k_{it}})k_{it}-\gamma_t k_{it+1}=0\ . \tag{89}$$

x_{it}: x; **k_{it}:** (1-δ)k; **k_{it+1}:** -γk; **γ_t:** -γk.

Now, dividing everything by k, we can write the log-linearized first order condition w.r.t. λ_i as:

$$-\gamma\hat{k}_{it+1}+(1-\delta_K)\hat{k}_{it}=-r_{xk}\hat{x}_{it}+\gamma\hat{\gamma}_t\ . \tag{90}$$

l) First order conditions w.r.t. μ_{1t}:

$$(1-\delta_H)h_t+B_{1t}(1-v_{1t})^{s_{bk}}k_{1t}^{s_{bk}}e_{1t}^{s_{bh}}h_t^{\chi}-\gamma_t h_{t+1}=0\ . \tag{91}$$

B_{1t}: z; **e_{1t}:** $s_{bh}z$; **v_{1t}:** $-s_{bk}\varphi z$; **k_{1t}:** $s_{bk}z$; **h_t:** $(1-\delta_H)h+\chi z$; **h_{t+1}:** $-\gamma h$; **γ_t:** $-\gamma h$.

Dividing everything by z, we can write the log-linearized f.o.c. w.r.t. μ_1 as:

$$-\frac{\gamma}{\gamma-1+\delta_H}\hat{h}_{t+1}+s_{bk}\hat{k}_{1t}+[\frac{1-\delta_H}{\gamma-1+\delta_H}+\chi]\hat{h}_t=-s_{bh}\hat{e}_{1t}+s_{bk}\varphi\hat{v}_{1t}+\frac{\gamma}{\gamma-1+\delta_H}\hat{\gamma}_t-\hat{B}_{1t}\ . \tag{92}$$

7 Linearized matrices.

In our log-linearized system we have 10 control and *control-like* variables (C), 7 endogenous *state-like* and costate variables (S) and 4 exogenous state variables (Z), and we can collect them in the following vectors:

$$C_t = [\hat{n}_{1t}, \hat{n}_{2t}, \hat{e}_{1t}, \hat{e}_{2t}, \hat{v}_{1t}, \hat{v}_{2t}, \hat{x}_{1t}, \hat{x}_{2t}, \hat{nx}_t, \hat{\gamma}_t]^T ,$$
$$S_t = [\hat{k}_{1t}, \hat{k}_{2t}, \hat{h}_t, \hat{\lambda}_{1t}, \hat{\lambda}_{2t}, \hat{\mu}_{1t}, \hat{\mu}_{2t}]^T ,$$
$$Z_t = [\hat{A}_{1t}, \hat{A}_{2t}, \hat{B}_{1t}, \hat{B}_{2t},]^T .$$

The 10 log-linearized first order conditions for the control variables can be summarized in the following linear sub-system:

$$M_{cc} C_t = M_{cs} S_t + M_{cz} Z_t , \tag{93}$$

In the same way, the 7 log-linearized first order conditions for the endogenous state variables can be summarized into:

$$M_{ss}(L) S_{t+1} = M_{sc}(L) C_{t+1} + M_{sz}(L) Z_{t+1} , \tag{94}$$

where $M_{ss}(L)$, $M_{sc}(L)$ and $M_{sz}(L)$ are matrix polynomials in the back shift operator L at most of power 1. The previous sub-system can also be written as:

$$[M_{ss}(0) + M_{ss}(1) \cdot L] S_{t+1} = [M_{sc}(0) + M_{sc}(1) \cdot L] C_{t+1} + [M_{sz}(0) + M_{sz}(1) \cdot L] Z_{t+1} . \tag{95}$$

where $M_{ss}(0)$, $M_{sc}(0)$ and $M_{sz}(0)$ contain the constant terms of the polynomials, while $M_{ss}(1)$, $M_{sc}(1)$ and $M_{sz}(1)$ contain the elements that multiply L.

8 Auxiliary variables.

Beside the variables listed in (93), there are other 12 variables of interest that we would like to recover. From the following expressions for normalized output in consumption good sector, leisure, normalized human capital stock in country 2, normalized consumption, normalized investments in human capital, normalized basic savings, normalized true savings, and standard "Solow" residuals:

$$y_{1t} = A_{1t} v_{1t}^{s_{ak}} k_{1t}^{s_{ak}} n_{1t}^{s_{ah}} h_t^{\zeta} , \qquad y_{2t} = A_{2t} v_{2t}^{s_{ak}} k_{2t}^{s_{ak}} n_{2t}^{s_{ah}} (1 - h_t)^{\zeta} , \tag{96}$$

$$l_{it} = 1 - n_{it} - e_{it} , \qquad h_{2t} = 1 - h_t \tag{97}$$

$$c_{1t} = A_{1t} v_{1t}^{s_{ak}} k_{1t}^{s_{ak}} n_{1t}^{s_{ah}} h_t^{\zeta} - x_{1t} - n x_t \ , \qquad c_{2t} = A_{2t} v_{2t}^{s_{ak}} k_{2t}^{s_{ak}} n_{2t}^{s_{ah}} (1-h_t)^{\zeta} - x_{2t} + \pi n x_t \ , \tag{98}$$

$$z_{1t} = B_{1t}(1-v_{1t})^{s_{bk}} k_{1t}^{s_{bk}} e_{1t}^{s_{bh}} h_t^{\zeta} \ , \qquad z_{2t} = B_{2t}(1-v_{2t})^{s_{bk}} k_{2t}^{s_{bk}} e_{2t}^{s_{bh}} (1-h_t)^{\zeta} \ , \tag{99}$$

$$bs_{1t} = x_{1t} + n x_t \ , \qquad bs_{2t} = x_{2t} - \pi n x_t \ , \qquad ts_t = \sum_{i=1}^{2} \Pi_i x_{it} \ , \tag{100}$$

$$a_{1t} = A_{1t} v_{1t}^{s_{ak}} k_{1t}^{s_{ak}} h_t^{\zeta} \ , \qquad a_{2t} = A_{2t} v_{2t}^{s_{ak}} k_{2t}^{s_{ak}} (1-h_t)^{\zeta} \ , \tag{101}$$

we can compute their approximated value near the steady-state:

$$\begin{aligned} \hat{y}_{1t} &= \hat{A}_{1t} + s_{ak}\hat{v}_{1t} + s_{ak}\hat{k}_{1t} + s_{ah}\hat{n}_{1t} + \zeta \hat{h}_t \ , \\ \hat{y}_{2t} &= \hat{A}_{2t} + s_{ak}\hat{v}_{2t} + s_{ak}\hat{k}_{2t} + s_{ah}\hat{n}_{2t} - \zeta \hat{h}_t \ , \end{aligned} \tag{102}$$

$$\begin{aligned} s_c\hat{c}_{1t} &= \hat{A}_{1t} + s_{ak}\hat{v}_{1t} + s_{ak}\hat{k}_{1t} + s_{ah}\hat{n}_{1t} + \zeta \hat{h}_t - s_x\hat{x}_{1t} - n\hat{x}_{it} \ , \\ s_c\hat{c}_{2t} &= \hat{A}_{2t} + s_{ak}\hat{v}_{2t} + s_{ak}\hat{k}_{2t} + s_{ah}\hat{n}_{2t} - \zeta \hat{h}_t - s_x\hat{x}_{2t} + \pi n\hat{x}_t \ , \end{aligned} \tag{103}$$

$$\hat{z}_{1t} = \hat{B}_{1t} - s_{bk}\varphi\hat{v}_{1t} + s_{bk}\hat{k}_{1t} + s_{bh}\hat{e}_{1t} + \zeta \hat{h}_t \ , \tag{104}$$

$$\hat{z}_{2t} = \hat{B}_{2t} - s_{bk}\varphi\hat{v}_{2t} + s_{bk}\hat{k}_{2t} + s_{bh}\hat{e}_{2t} - \zeta \hat{h}_t \ , \tag{105}$$

$$s_x\hat{bs}_{1t} = s_x\hat{x}_{1t} + n\hat{x}_t \ , \qquad s_x\hat{bs}_{2t} = s_x\hat{x}_{2t} - \pi n\hat{x}_t \ , \qquad \hat{ts}_t = \sum_{i=1}^{2} \Pi_i \hat{x}_{it} \ , \tag{106}$$

$$\hat{a}_{1t} = \hat{A}_{1t} + s_{ak}\hat{v}_{1t} + s_{ak}\hat{k}_{1t} + \zeta \hat{h}_t \ , \qquad \hat{a}_{2t} = \hat{A}_{2t} + s_{ak}\hat{v}_{2t} + s_{ak}\hat{k}_{2t} - \zeta \hat{h}_t \ . \tag{107}$$

We can write the previous system in a more compact form as:

$$F_t = FV_c C_t + FV_{kz} KZ_t + FV_l L_t \ , \tag{108}$$

where:

$$F_t \equiv [\hat{y}_{1t}\ \ \hat{y}_{2t}\ \ \hat{c}_{1t}\ \ \hat{c}_{2t}\ \ \hat{l}_{1t}\ \ \hat{l}_{2t}\ \ \hat{z}_{1t}\ \ \hat{z}_{2t}\ \ \hat{bs}_{1t}\ \ \hat{bs}_{2t}\ \ \hat{ts}_t\ \ \hat{h}_{2t}\ \ \hat{a}_{1t}\ \ \hat{a}_{2t}]^T ,$$

$$KZ_t \equiv [\hat{k}_{1t}\ \ \hat{k}_{2t}\ \ \hat{h}_t\ \ \hat{A}_{1t}\ \ \hat{A}_{2t}\ \ \hat{B}_{1t}\ \ \hat{B}_{2t}]^T ,$$

$$L_t \equiv [\hat{\lambda}_{1t}\ \ \hat{\lambda}_{2t}\ \ \hat{\mu}_{1t}\ \ \hat{\mu}_{2t}]^T .$$

9 System matrices.

$$M_{cc}=\begin{bmatrix}
-s_{ah}-s_c(s_{ak}+\omega) & 0 & -s_c\varsigma & 0 & -s_{ak}s_x & 0 & s_x & 0 & 1 & 0\\
0 & -s_{ah}-s_c(s_{ak}+\omega) & 0 & -s_c\varsigma & 0 & -s_{ak}s_x & 0 & s_x & -\pi & 0\\
s_c\xi_{ll}\omega-\xi_{lc}s_{ah} & 0 & s_c(\xi_{ll}\varsigma-s_{bk}) & 0 & -\xi_{lc}s_{ak}-s_c s_{bk}\varphi & 0 & \xi_{lc}s_x & 0 & \xi_{lc} & 0\\
0 & s_c\xi_{ll}\omega-\xi_{lc}s_{ah} & 0 & s_c(\xi_{ll}\varsigma-s_{bk}) & 0 & -\xi_{lc}s_{ak}-s_c s_{bk}\varphi & 0 & \xi_{lc}s_x & -\pi\xi_{lc} & 0\\
\xi_{cc}s_{ah}+s_c(s_{ah}-\xi_{cl}\omega) & 0 & -s_c(\xi_{cl}\varsigma+s_{bh}) & 0 & \xi_{cc}s_{ak}-s_c(s_{ah}+s_{bh}\varphi) & 0 & -\xi_{cc}s_x & 0 & -\xi_{cc} & 0\\
0 & \xi_{cc}s_{ah}+s_c(s_{ah}-\xi_{cl}\omega) & 0 & -s_c(\xi_{cl}\varsigma+s_{bh}) & 0 & \xi_{cc}s_{ak}-s_c(s_{ah}+s_{bh}\varphi) & 0 & -\xi_{cc}s_x & \pi\xi_{cc} & 0\\
s_c\xi_{cl}\omega-\xi_{cc}s_{ah} & 0 & s_c\xi_{cl}\varsigma & 0 & -\xi_{cc}s_{ak} & 0 & \xi_{cc}s_x-s_c\xi_\psi & 0 & \xi_{cc} & 0\\
0 & s_c\xi_{cl}\omega-\xi_{cc}s_{ah} & 0 & s_c\xi_{cl}\varsigma & 0 & -\xi_{cc}s_{ak} & 0 & \xi_{cc}s_x-s_c\xi_\psi & -\pi\xi_{cc} & 0\\
s_c\xi_{cl}\omega-\xi_{cc}s_{ah} & \xi_{cc}s_{ah}-s_c\xi_{cl}\omega & s_c\xi_{cl}\varsigma & -s_c\xi_{cl}\varsigma & -\xi_{cc}s_{ak} & \xi_{cc}s_{ak} & \xi_{cc}s_x & -\xi_{cc}s_x & \dfrac{\xi_{cc}}{\Pi_2} & 0\\
0 & 0 & s_{bh} & s_{bh} & -s_{bk}\varphi & -s_{bk}\varphi & 0 & 0 & 0 & -\dfrac{\gamma}{z}
\end{bmatrix}.$$

$$M_{cs}=\begin{bmatrix}
s_{ak}s_x & 0 & \zeta s_x & 0 & 0 & 0 & 0\\
0 & s_{ak}s_x & -\zeta s_x & 0 & 0 & 0 & 0\\
\xi_{lc}s_{ak}-s_c s_{bk} & 0 & \xi_{lc}\zeta-s_c\chi & 0 & 0 & -s_c & 0\\
0 & \xi_{lc}s_{ak}-s_c s_{bk} & -(\xi_{lc}\zeta-s_c\chi) & 0 & 0 & 0 & -s_c\\
s_c(s_{bk}-s_{ak})-\xi_{cc}s_{ak} & 0 & s_c(\chi-\zeta)-\xi_{cc}\zeta & 0 & 0 & s_c & 0\\
0 & s_c(s_{bk}-s_{ak})-\xi_{cc}s_{ak} & -(s_c(\chi-\zeta)-\xi_{cc}\zeta) & 0 & 0 & 0 & s_c\\
\xi_{cc}s_{ak}-s_c\xi_\psi & 0 & \xi_{cc}\zeta & -s_c & 0 & 0 & 0\\
0 & \xi_{cc}s_{ak}-s_c\xi_\psi & -\xi_{cc}\zeta & 0 & -s_c & 0 & 0\\
\xi_{cc}s_{ak} & -\xi_{cc}s_{ak} & 0 & 0 & 0 & 0 & 0\\
-s_{bk} & -s_{bk} & 0 & 0 & 0 & 0 & 0
\end{bmatrix},\quad
M_{ce}=\begin{bmatrix}
s_x & 0 & 0 & 0\\
0 & s_x & 0 & 0\\
\xi_{lc} & 0 & -s_c & 0\\
0 & \xi_{lc} & 0 & -s_c\\
-(\xi_{cc}+s_c) & 0 & s_c & 0\\
0 & -(\xi_{cc}+s_c) & 0 & s_c\\
\xi_{cc} & 0 & 0 & 0\\
0 & \xi_{cc} & 0 & 0\\
\xi_{cc} & -\xi_{cc} & 0 & 0\\
0 & 0 & -1 & -1
\end{bmatrix}.$$

$$M_{ss}0 = \begin{bmatrix}
\xi_{cc}s_{ak} - s_c(s_{ah} + \frac{s_{bh}}{\varphi}) - \xi_\psi r_{xk}\Delta & 0 & \xi_{cc}\zeta + s_c(\zeta + \frac{\chi}{\varphi}) & (1-\delta_K)\Delta & 0 & \frac{s_c}{\varphi} & 0 \\
0 & \xi_{cc}s_{ak} - s_c(s_{ah} + \frac{s_{bh}}{\varphi}) - \xi_\psi r_{xk}\Delta & -[\xi_{cc}\zeta + s_c(\zeta + \frac{\chi}{\varphi})] & 0 & (1-\delta_K)\Delta & 0 & \frac{s_c}{\varphi} \\
\xi_{cc}s_{ak} + s_c(s_{ak} + \frac{s_{bk}}{\upsilon}) & 0 & \xi_{cc}\zeta - s_c(1 - \zeta + \frac{1-\chi}{\upsilon}) & 0 & 0 & (1-\delta_H)\Theta + \frac{s_c}{\upsilon} & 0 \\
0 & \xi_{cc}s_{ak} + s_c(s_{ak} + \frac{s_{bk}}{\upsilon}) & -[\xi_{cc}\zeta - s_c(1 - \zeta + \frac{1-\chi}{\upsilon})] & 0 & 0 & 0 & (1-\delta_H)\Theta + \frac{s_c}{\upsilon} \\
-\gamma & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\gamma & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{\gamma}{\gamma - 1 + \delta_H} & 0 & 0 & 0 & 0
\end{bmatrix}.$$

$$M_{ss}1 = \begin{bmatrix}
0 & 0 & 0 & -\frac{\gamma^{-\xi_{cc}}}{\beta}\Delta & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{\gamma^{-\xi_{cc}}}{\beta}\Delta & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{\gamma^{-\xi_{cc}}}{\beta}\Theta & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\frac{\gamma^{-\xi_{cc}}}{\beta}\Theta \\
1-\delta_K & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1-\delta_K & 0 & 0 & 0 & 0 & 0 \\
s_{bk} & 0 & \frac{1-\delta_H}{\gamma - 1 + \delta_H} + \chi & 0 & 0 & 0 & 0 \quad 0
\end{bmatrix}.$$

$$\begin{matrix} M_{sc}0 = \\ M_{sc}1 = \end{matrix} \begin{bmatrix}
0 & 0 & s_c(\xi_{cl}\omega - s_{ah}) & \xi_{cc}s_{ah} & 0 & 0 & 0 & 0 & 0 & -\xi_{cc}(\xi_{cl}\varsigma - \frac{s_{bh}}{\varphi}) \;\; -\frac{\gamma^{-\xi_{cc}}}{\beta}\Delta & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & s_c(\xi_{cl}\omega - s_{ah}) & -\xi_{cc}s_{ah} & 0 & -\xi_{cc}\frac{\gamma^{-\xi_{cc}}}{\beta}\Delta & 0 \\
0 & 0 & s_c(\xi_{cl}\omega - s_{ah}) & \xi_{cc}s_{ah} & 0 & 0 & 0 & 0 & 0 & -\xi_{cc}(\xi_{cl}\varsigma - \frac{s_{bh}}{\upsilon}) \;\; -\frac{\gamma^{-\xi_{cc}}}{\beta}\Theta & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & s_c(\xi_{cl}\omega - s_{ah}) & -\xi_{cc}s_{ah} & 0 & -\xi_{cc}\frac{\gamma^{-\xi_{cc}}}{\beta}\Theta & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -r_{xk} & 0 & 0 & 0 & \gamma \quad 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -r_{xk} & 0 & 0 & \gamma \quad 0 \\
0 & 0 & -s_{bh} & 0 & s_{bk}\varphi & 0 & 0 & 0 & 0 & \frac{\gamma}{\gamma - 1 + \delta_H} & 0
\end{bmatrix}, \quad
M_{se}0 = \begin{bmatrix}
0 & s_c(s_{bk} - s_{ak}) - \xi_{cc}s_{ak} & 0 \\
s_c(\xi_{cl}\varsigma - \frac{s_{bh}}{\varphi}) & 0 & s_c(s_{bk} - s_{ak}) - \xi_{cc}s_{ak} \\
0 & s_c(s_{bk}\frac{\varphi}{\upsilon} - s_{ak}) - \xi_{cc}s_{ak} & 0 \\
s_c(\xi_{cl}\varsigma - \frac{s_{bh}}{\upsilon}) & 0 & s_c(s_{bk}\frac{\varphi}{\upsilon} - s_{ak}) - \xi_{cc}s_{ak} \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
-(\xi_{cc} + s_c) & 0 & -\frac{s_c}{\varphi} & 0 \\
0 & -(\xi_{cc} + s_c) & 0 & -\frac{s_c}{\varphi} \\
-(\xi_{cc} + s_c) & 0 & -\frac{s_c}{\upsilon} & 0 \\
0 & -(\xi_{cc} + s_c) & 0 & -\frac{s_c}{\upsilon} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}, \quad
M_{se}1 = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & \xi_{cc}s_x & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0
\end{bmatrix}.$$

$$\begin{bmatrix}
0 & \xi_{cc}s_x - \xi_\psi r_{xk}\Delta & 0 & \xi_{cc} & 0 \\
0 & 0 & \xi_{cc}s_x - \xi_\psi r_{xk}\Delta & -\xi_{cc}\pi & 0 \\
0 & 0 & 0 & \xi_{cc} & 0 \\
0 & 0 & \xi_{cc}s_x & -\xi_{cc}\pi & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}.$$

$$FV_c = \begin{bmatrix} s_{ah} & 0 & 0 & 0 & s_{ak} & 0 & 0 & 0 & 0 & 0 \\ 0 & s_{ah} & 0 & 0 & 0 & s_{ak} & 0 & 0 & 0 & 0 \\ \frac{s_{ah}}{s_c} & 0 & 0 & 0 & \frac{s_{ak}}{s_c} & 0 & -\frac{s_x}{s_c} & 0 & -\frac{1}{s_c} & 0 \\ 0 & \frac{s_{ah}}{s_c} & 0 & 0 & 0 & \frac{s_{ak}}{s_c} & 0 & -\frac{s_x}{s_c} & \frac{\pi}{s_c} & 0 \\ -\omega & 0 & -\varsigma & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\omega & 0 & -\varsigma & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & s_{bh} & 0 & -s_{bk}\varsigma & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & s_{bh} & 0 & -s_{bk}\varsigma & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \frac{1}{s_x} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \frac{\pi}{s_x} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \Pi_1 & \Pi_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & s_{ak} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & s_{ak} & 0 & 0 & 0 & 0 \end{bmatrix}, \quad FV_{ke} = \begin{bmatrix} s_{ak} & 0 & \zeta & 1 & 0 & 0 & 0 \\ 0 & s_{ak} & -\zeta & 0 & 1 & 0 & 0 \\ \frac{s_{ak}}{s_c} & 0 & \frac{\zeta}{s_c} & \frac{1}{s_c} & 0 & 0 & 0 \\ 0 & \frac{s_{ak}}{s_c} & -\frac{\zeta}{s_c} & 0 & \frac{1}{s_c} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ s_{bk} & 0 & \zeta & 0 & 0 & 1 & 0 \\ 0 & s_{bk} & -\zeta & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ s_{ak} & 0 & \zeta & 1 & 0 & 0 & 0 \\ 0 & s_{ak} & -\zeta & 0 & 1 & 0 & 0 \end{bmatrix}, \quad FV_l = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

10 Solution procedure.

The solution procedure outlined in the next pages follows King, Plosser and Rebelo (1987) very closely. If the matrix M_{cc} is invertible, as it should be in our case, we can solve (103) for C_t and substitute the result in (104), getting:

$$[M_{ss}^*(0) + M_{ss}^*(1)L]S_{t+1} = [M_{sz}^*(0) + M_{sz}^*(1)L]Z_{t+1} \ , \tag{109}$$

where:

$$\begin{aligned} M_{ss}^*(0) &\equiv M_{ss}(0) - M_{sc}(0)Q_{cs} \ , \quad & M_{ss}^*(1) &\equiv M_{ss}(1) - M_{sc}(1)Q_{cs} \ , \\ M_{sz}^*(0) &\equiv M_{sz}(0) + M_{sc}(0)Q_{cz} \ , \quad & M_{sz}^*(1) &\equiv M_{sz}(1) + M_{sc}(1)Q_{cz} \ . \end{aligned} \tag{110}$$

and $Q_{cs} \equiv M_{cc}^{-1}M_{cs}$, $Q_{cz} \equiv M_{cc}^{-1}M_{cz}$. If the matrix $M_{ss}^*(0)$ is invertible, we may rewrite the system as:

$$S_{t+1} = WS_t + RZ_{t+1} + QZ_t \ , \tag{111}$$

where:

$$W \equiv -[M_{ss}^*(0)]^{-1}M_{ss}^*(1) \ , \quad R \equiv [M_{ss}^*(0)]^{-1}M_{sz}^*(0) \ , \quad Q \equiv [M_{ss}^*(0)]^{-1}M_{sz}^*(1) \ . \tag{112}$$

Under the King, Plosser and Rebelo (1987) certainty equivalence assumption, to switch from the deterministic case to the stochastic one we need simply to rewrite (111) in expectations:

$$E(S_{t+1}) = WS_t + RE(Z_{t+1}) + QZ_t \ , \tag{113}$$

where $E(\cdot)$ is the expectation operator evaluated at time t. Given that $E(Z_{t+1}) = \rho Z_t$, we can rewrite (113) as:

$$E(S_{t+1}) = WS_t + AZ_t \ , \tag{114}$$

where $A \equiv R\rho + Q$. If P is the modal matrix of W and μ its canonical form (with the eigenvalues on the diagonal ordered in ascending absolute value), and if P is invertible, we may decompose W as $P\mu P^{-1}$. Lets furthermore partition the matrices W, μ, P^{-1} and A in the following way:

$$W \equiv \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix}, \quad P^{-1} \equiv \begin{bmatrix} p_{11}^{*} & p_{12}^{*} \\ p_{21}^{*} & p_{22}^{*} \end{bmatrix}, \quad \mu \equiv \begin{bmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{bmatrix}, \quad A \equiv \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}. \tag{115}$$

As usual, the dynamics of the system (114) are governed by the eigenvalues of W. Assuming that the first four eigenvalues are unstable and the last four stable, the system will be saddle-point stable and there will be only one initial vector of shadow prices compatible with the transversality conditions. Pre-multiplying (114) by P^{-1}, we transformed the original system in a transformed system comprised of two decoupled vectors of difference equations (μ is diagonal):

$$E(\tilde{S}_{t+1}) = \mu \tilde{S}_t + B Z_t \ , \tag{116}$$

where $\tilde{S}_t = P^{-1} S_t$ and $B \equiv P^{-1} A$. The capital component of the transformed system is:

$$E(\tilde{K}_{t+1}) = \mu_1 \tilde{K}_t + b_1 Z_t \ , \tag{117}$$

where $K_t \equiv [\hat{k}_{1t}, \hat{k}_{2t}, \hat{h}_t]^T$ and b_1 is implicitly defined by $B \equiv [b_1 | b_2]$. Since the elements on the diagonal of μ_1 are less than one in absolute value, the sub-system (114) is stable in the forward direction; furthermore, since all the variables in K_t are predetermined, the initial condition K_0 completely determines the solution.
Unfortunately, the analogous sub-system for the shadow prices component:

$$E(\tilde{L}_{t+1}) = \mu_2 \tilde{L}_t + b_2 Z_t \ , \tag{118}$$

where $L_t \equiv [\hat{\lambda}_{1t}, \hat{\lambda}_{2t}, \hat{\mu}_{1t}, \hat{\mu}_{2t}]^T$, is stable in the backward direction, since the elements of μ_2 exceed one in absolute value. This means that it is necessary to impose a terminal rather than an initial condition for the transformed shadow prices. Rewrite (118) as:

$$\tilde{L}_t = \mu_2^{-1} E(\tilde{L}_{t+1}) + \mu_2^{-1} b_2 Z_t = d_1 E(\tilde{L}_{t+1}) + d_2 Z_t \ , \tag{119}$$

The solution to (119) is given by:

$$\tilde{L}_t = - \sum_{k=0}^{\infty} d_1^k d_2 E_t(Z_{t+k}) = - \sum_{k=0}^{\infty} d_1^k d_2 \rho^k Z_t = L_{ee} Z_t \ . \tag{120}$$

Applying the *vec* ($\vec{x}$) operator to L_{ee} we get:

$$\vec{L}_{ee} = -(I - \rho' \otimes d_1)^{-1} \vec{d}_2 \ . \tag{121}$$

The relationships between the transformed variables and the original ones are:

$$\tilde{K}_t = p_{11}^* K_t + p_{12}^* L_t \ , \qquad \tilde{L}_t = p_{21}^* K_t + p_{22}^* L_t \ . \tag{122}$$

From the second element of (116), we get:

$$L_t = p_{22}^{*\,-1} \tilde{L}_t - p_{22}^{*\,-1} p_{21}^* K_t \ . \tag{123}$$

or:

$$L_t = L_k K_t + L_z Z_t = L_{kz} KZ_t \ , \tag{124}$$

where $L_z \equiv p_{22}^{*\,-1} L_{ee}$, $L_k \equiv -p_{22}^{*\,-1} p_{21}^*$ and $L_{kz} \equiv [L_k \ L_z]$.
From (142) we can isolate the capital component of the original system, given by:

$$E(K_{t+1}) = w_{11} K_t + w_{12} L_t + a_1 Z_t \ . \tag{125}$$

Since all variables in the K_t vector are predetermined in the Blanchard-Kahn sense (expectational error equal to zero), we can rewrite (123) as:

$$K_{t+1} = w_{11} K_t + w_{12} L_t + a_1 Z_t \ . \tag{126}$$

Taking into account (124), we can rearrange (125) as:

$$K_{t+1} = (w_{11} + w_{12} L_k) K_t + (w_{12} L_z + a_1) Z_t \ . \tag{127}$$

Combining (127) and (8) we get:

$$KZ_{t+1} = M_{kz} KZ_t + e_t \ , \tag{128}$$

where:

$$M_{kz} \equiv \begin{bmatrix} (w_{11} + w_{12}L_k) & (w_{12}L_z + a_1) \\ 0 & \rho \end{bmatrix}, \qquad e_t \equiv \begin{bmatrix} 0 \\ \varepsilon_t \end{bmatrix}. \tag{129}$$

From (94) we have that:

$$C_t = Q_{cs}\begin{bmatrix} K_t \\ L_{kz}KZ_t \end{bmatrix} + Q_{cz}Z_t \ . \tag{130}$$

Partitioning the matrix Q_{cs} we may rewrite (130) as:

$$C_t = \left[[Q_{cs}^k \ Q_{cz}] + Q_{cs}^l L_{kz} \right] KZ_t = C_{kz}KZ_t \ . \tag{131}$$

Turning now to the auxiliary variables, we can substitute (124) and (131) into (107) to get:

$$F_t = (FV_c C_{kz} + FV_{kz} + FV_l L_{kz})KZ_t = F_{kz}KZ_t \ . \tag{132}$$

Equations (128), (130) and (132) are the key elements that permit the numerical simulation of our linearized system: given the realization of the exogenous shocks, the state variables evolve according to (128), while the control variables evolve according to (130) and the auxiliary ones evolve according to (132).
For simplicity, we may define a new vector $H_t \equiv [C_t \,|\, F_t]$ such that:

$$H_t = H_{kz}KZ_t \ , \tag{133}$$

where $H_{kz} \equiv [C_{kz} | F_{kz}]$.
As far as the population moments are concerned, from (128) we easily get:

$$\Sigma_{kz} = M_{kz}\Sigma_{kz}M_{kz}' + \Sigma_e \ , \tag{134}$$

or:

$$\vec{\Sigma}_{kz} = (I - M_{kz}' \otimes M_{kz}')^{-1}\vec{\Sigma}_e \ . \tag{135}$$

It is also immediate to show that:

$$\Sigma_H = H_{kz}\Sigma_{kz}H_{kz}' \ . \tag{136}$$